Essays on Social Learning and Reputation Building

Dissertation

submitted to the

Faculty of Business, Economics and Informatics

of the University of Zurich

to obtain the degree of

Doktor der Wirtschaftswissenschaften, Dr. oec.

(corresponds to Doctor of Philosophy, PhD)

presented by

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The Faculty of Business, Economics and Informatics of the University of Zurich hereby authorizes the printing of this dissertation, without indicating an opinion of the views expressed in the work.

Zurich, 15.07.2020

Chairman of the Doctoral Board: Prof. Dr. Steven Ongena

Acknowledgements

I was fortunate to have advisors who were willing to invest their time and efforts into me. Without the kind support from Armin Schmutzler and Marek Pycia the quality of my research projects and the job market outcome would be significantly lower.

Given the chance I would like to acknowledge contribution to my development from professors, fellow Ph.D. students and staff from the University of Zurich. In particular I would like to thank Christian Ewerhart, Andreas Hefti, Nick Netzer, Jakub Steiner, Hans Joachim Voth, Jean-Michel Benkert, Stefanie Bossard, Shangen Li, Shuo Liu, Kremena Valkanova, Mirjam Britschgi and Karin Wyss.

I am also thankful to my longtime co-author Egor Starkov.

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Part I

Dissertation Overview

Dissertation Overview

This dissertation consists of four separate research papers which are centered around two themes - social learning and reputation building.

In the first paper "Experts, Quacks and Fortune-Tellers: Dynamic Cheap Talk with Career Concerns", Egor Starkov and I study markets for forecasters who's well-being is primarily determined by their public reputation. The development of social media, public blogs, and YouTube channels has dramatically increased the amount of viewpoints available to the public about any major topic or event. This is however not necessarily beneficial. The prospect of media attention, fame and associated benefits potentially attracts lots of quacks who broadcast their thoughts and opinions on who will win elections, what stock to buy and whether a next economic crisis will hit the economy in the upcoming year even if they have no insight into the topic. Their presence thus makes the public suspicious about the conveyed information: did it come from an expert's well-founded analysis of a situation, or is it just a quack's random guess? This harms both sides of the market. On the one hand, it means that the public cannot distinguish justified forecasts from quacks' random utterances. On the other hand, it makes it difficult for real experts to signal their competence and gain the deserved reputation.

In our model, a forecaster who is privately aware of his competence, makes a choice of whether and when to make a prediction about the outcome of some future event (i.e. elections outcome, BTC/USD exchange rate or oil price). A competent expert *may* obtain some private knowledge about the outcome before it is revealed, while a quack never does. The forecaster only cares about belief about his competence: he collects per-period reputational benefits as before the outcome is revealed as after it.

We show that even in the absence of any explicit intertemporal trade-offs (such as higher precision of later information or preemption incentives in competitive settings), beliefs evolve in such a way that later predictions are less informative than earlier ones. This is because early reports are only ever made in equilibrium if some competent experts reveal their private information early on. To incentivize them to do so, the value of information should be maximal at that time – i.e., it should be weakly decreasing over time. We show that the value of information is directly related to the informativeness of the report, which gives the result. The second result of the paper is that any report hurts the forecaster's reputation in the short run, meaning that all possible benefits from making a prediction are concentrated in the ex post reputation premium for having guessed the state correctly.

A more surprising conclusion is that predictions, although considered informative, are received with solid skepticism in the sense that the forecaster's reputation drops once he makes a prediction. Silent forecasters see their reputation gradually improving. Those, on the other hand, who choose to make a prediction and take a hit to their reputation, are gambling for the grand prize which is the reputation bonus for anyone who guesses the outcome correctly. The incentives to gamble on reputation are present even for quacks, which may appear suspicious given that they would otherwise see their reputation improving even when they stay silent. This is explained by our finding that informative equilibria exist only if gains from reputation are "sufficiently convex" in belief about forecaster's competence. Otherwise only trivial "babbling" equilibria exist, where predictions, if they take place, convey absolutely no information about either the outcome of the event or the competence of the forecaster.

The second research paper can be attributed to both subtopics of the dissertation. Online product reviews, through which consumers communicate their experiences constitute an important channel for learning the quality of a good or a service. This paper explores the mechanism through which sellers can undermine this channel, which is censorship. In many circumstances, producers can remove unfavorable reviews of their own product and leave only positive ones, thus hindering the transmission of information to consumers.

A straightforward conjecture would be that no meaningful bad reviews ever remain, and those that do convey absolutely no information. This is because the seller would delete any review that harms sales. However, in practice we observe plenty of informative bad reviews even when censorship opportunities exist. Furthermore, previous research suggests that a small number of bad reviews may in fact improve product reputation, leading to higher sales. Why might the seller be willing *not* to censor unfavorable reviews? Furthermore, how is the informational content of such reviews affected by censorship? This paper answer these questions.

It shows that if some consumers in the market are naive in the sense of being unaware of possible censorship, then no bad reviews should actually be perceived as *bad* by a rational consumer. More importantly, if market share of naive consumers is positive but not too large then bad reviews are actually *good*. In other words, any rational consumer *improves* her belief about the product quality upon observing any review that says the product is *bad*. This is because the additional signal contained in the fact that a bad review has slipped through the censorship machine outweighs the face value of a review. What truly harms sales in this situation is the absence of reviews.

The paper constructs a model, in which a long-lived seller offers a good of privately known quality to a sequence of short-lived consumers. Consumption utility is suggestive about the product quality and is relayed to future consumers through reviews, which may be deleted by the seller. Consumers differ in their inference process given posted reviews: some are strategic – i.e. fully aware of the seller's censorship capabilities, – while others are naive and ignore censorship (or are merely unaware of it).

The reason that in all equilibria any revealed bad review has to improve seller's reputation is two-fold. The main idea is that it only makes sense for the seller to reveal a bad review if it does not have any detrimental effect on future sales. The primary channel that drives the effect can then be formulated as follows: if a bad review was published and it harms sales to naive consumers, then it should increase sales to rational consumers, as otherwise it would not have been allowed. In particular, this increase in sales is attained by improving firm's reputation in the eyes of rational consumers.

The secondary channel is more involved. When the seller gets close to losing naive consumers, the primary channel above creates suspense among rational consumers, who await the seller's next move. If he reveals a bad review, he receives a significant reputation premium, but the flip side is that rational consumers also lose faith quickly if no reviews are revealed. Revealing bad reviews earlier in the game thus creates more suspense – since it brings naive consumers closer to leaving the market, – and leads to faster alienation of rational consumers. Therefore, in order for the seller to have any incentive to reveal bad reviews, he should be compensated with a reputation premium for doing so. Notably, this premium is awarded even though he does not lose any naive consumers immediately.

The argument above explains why any bad review should improve seller's reputation but not how it is achieved, and neither it says whether any bad reviews *are* actually revealed in equilibrium. While deleting all reviews is always an equilibrium, we show that there exist other equilibria, in which bad reviews are revealed in a payoff-relevant way – i.e. it is profitable for the seller to reveal them. Moreover, it is more beneficial to do so for the seller with a high-quality product than with a low-quality product, which is exactly the mechanism that generates the result.

The third research paper explores how people transmit information when writing reviews. It first shows that, potentially surprisingly, transmitting own experience truthfully to subsequent consumers is neither an optimal "social norm" (i.e., it does not maximize aggregate social welfare), nor it is an equilibrium if consumers explicitly care about the welfare of other consumers when writing a review.

Egor Starkov and I consider a model in which every consumer purchases the durable good only if it is individually rational to do so and afterwards leaves a review to maximize the welfare of subsequent consumers. Social welfare is maximized by some degree of myopically suboptimal purchasing decisions, i.e., the option value of discovering a good product makes it socially optimal to induce a purchase that would not be individually optimal. This creates a conflict between a today's consumer who is writing a review and wants to induce further experimentation with a product, and tomorrow's consumer who reads this review and tries to understand whether buying the product is individually – rather than socially – rational.

This conflict adds noise to communication through reviews. Instead of reporting their experiences truthfully, consumers obfuscate their reviews to foster experimentation, which creates information losses. Despite the conflict arising only in a special set of circumstances – when the product is believed to be good enough to experiment with socially, but not good enough to buy for an individual, – we show that the effects of this conflict propagate and distort communication in other cases as well. In particular, we show that communication takes the interval structure known in the cheap talk literature, when senders with similar private beliefs pool on the same message.

Finally, the fourth research paper studies the question of strategic experimentation under the assumption that intensity of experimentation can not be changed immediately, but instead features some rigidity. Classical models of experimentation such as Bolton and Harris [1999] and Keller, Rady, and Cripps [2005] assume that intensity of experimentation in a given moment in time does not depend on the intensity of experimentation in previous moments. In contrast we provide a model where intensities are correlated between time periods. We fully solve for optimal experimentation strategy for a single experimenter problem, N-experimenters cooperative problem and N-experimenters strategic problem.

Part II

Research Papers

Timing of Predictions in Dynamic Cheap Talk: Experts vs. Quacks[1]

joint with Egor Starkov

Introduction

Where there is uncertainty, there are analysts – be it stock prices, macroeconomic trends, elections, or sports matches. Any major event summons numerous predictions of its outcome from people who claim to be experts in the field. However, not all of these predictions are necessarily backed by knowledge or understanding of the situation. This raises challenges for both sides of the predictions market. The expert forecasters must find a way to signal their competence through their predictions. The public, on the other hand, must find a way to identify informative predictions by the experts among the quacks' uninformative opinions.

This paper focuses on the *timing* of the forecaster's prediction as a signaling device. We explore the questions of how the choice of timing of the forecaster's prediction can signal their competence and how the amount of information about the state of the world contained in the predictions changes over time. To motivate the question, consider the case of 2016 US President Elections. Figure 1 demonstrates the ratings of presidential candidates Donald Trump and Hillary Clinton in a span of 9 month before the elections. Consider three different claims that Donald Trump will win, made in February 2016 (before Republican Party presidential primaries), May 2016, and September 2016 respectively.[2] Which of these predictions appears more credible ex post? Which of these predictions appeared more credible at the time? Would the authors of the early predictions have benefited from delaying it and, if yes, why did they not? Would the authors of the later predictions have made their predictions earlier if they had the information? These are the questions we attempt to answer in this paper.

We present a model of dynamic cheap talk with career concerns. In our model a forecaster, who is privately aware of his competence, makes a choice of whether and *when* to make a prediction about the outcome of some future exogenous event (state of the world). A competent forecaster (an

[1]This paper should be cited as A. Smirnov, E. Starkov. Timing of Predictions in Dynamic Cheap Talk: Experts vs. Quacks. mimeo, 2020.

[2]February 2016: `https://www.sbstatesman.com/2016/02/23/political-science-professor-forecasts-trump-as-general-e`
May 2016: `https://www.salon.com/2016/05/23/donald_trump_is_going_to_win_this_is_why_hillary_clinton_cant_defeat_what_trump_represents/`,
September 2016: `https://www.washingtonpost.com/news/the-fix/wp/2016/09/23/trump-is-headed-for-a-win-says-professor-whos-predicted-30-years-of-presidential-outcomes-correctly/`

expert) *may* have some private knowledge about the outcome, while an incompetent forecaster (a quack) never does. There is no direct conflict between the forecaster and the observer (the public) in our model: the forecaster only cares about his reputation, while the observer only cares about the information concerning the outcome. The conflict comes from within the forecasters market, with the quacks trying to blend in with the experts in pursuit of reputation (and benefits that high reputation grants), preventing experts from conveying valuable information to the public.

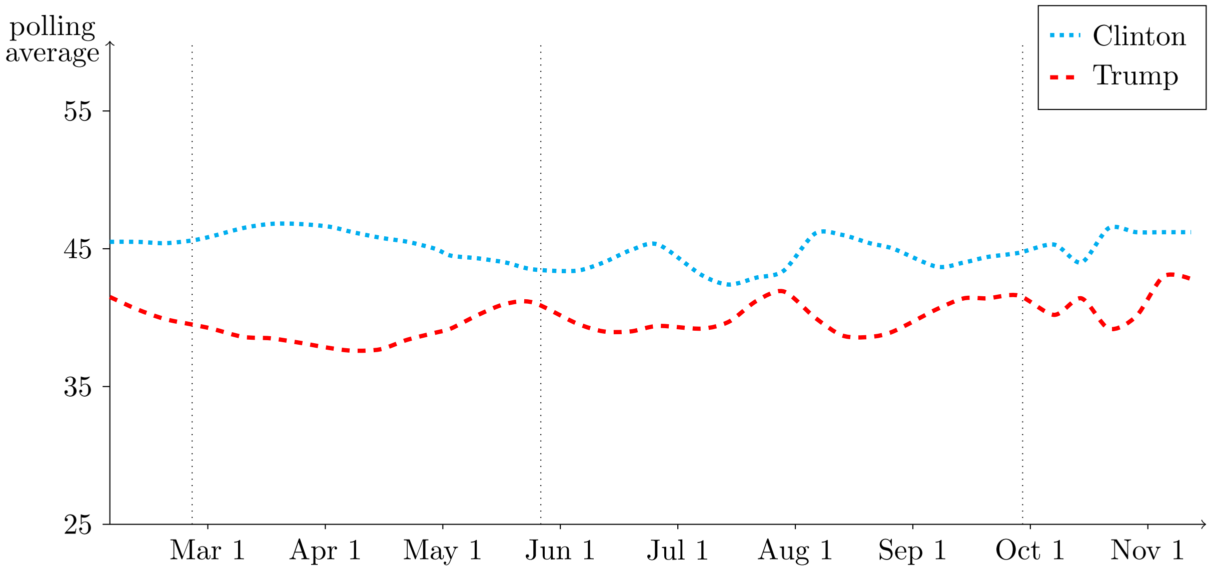

Figure 1: Donald Trump and Hillary Clinton polling averages before the 2016 elections. *Dotted verticals represent the dates of news articles mentioned in the text. Poll data retrieved from* `https://elections.huffingtonpost.com/pollster/2016-general-election-trump-vs-clinton`, *used under CC 3.0 BY-NC-SA license.*

We discover that this conflict between the quacks and the experts imposes a lot of structure on equilibrium outcomes. Our first finding is that in equilibrium, the later predictions are less informative than the earlier ones. Otherwise – if later predictions are more informative – they are also rewarded with higher reputation by the public, which would make the experts want to delay their reports. Therefore, the informativeness of predictions must deteriorate over time to incentivize the experts to reveal their information early, and to prevent a market shutdown when no predictions are made until the very final moment.

A more surprising finding of our paper is that *all* predictions in such equilibria, although considered informative, are received with solid scepticism by the public. This is in the sense that making any prediction drops the forecaster's reputation relative to what he could get by staying quiet. Thus silence is indeed golden in our model – silent forecasters see their reputation gradually improving. Those, on the other hand, who choose to make a prediction and take a hit to their reputation, are gambling for the grand prize that is the reputation bonus for predicting the outcome correctly.

A typical path of the forecaster's reputation arising from our model is illustrated in Figure 2. In this example the event occurs in period 6 and the forecaster starts with reputation b_0. The forecaster makes his report in period 4, and until then his reputation gradually increases. After the report his reputation drops until the event outcome is revealed, at which point he receives a reputation premium if his prediction turned out correct and is penalized by low reputation otherwise.

Given everything said above, it is not obvious why a forecaster would ever prefer to make any prediction, i.e., take a risky gamble at the cost of short-run reputation, when staying silent would yield a risk-free high reputation. As we show, equilibria of the form above only exist if forecasters are sufficiently risk-loving or, alternatively, if gains from reputation are sufficiently convex – i.e., if the gamble of making a report is appealing enough to the quack. Whenever this is not the case, only "static" equilibria exist, in which all reports are made at some single predetermined date.[3]

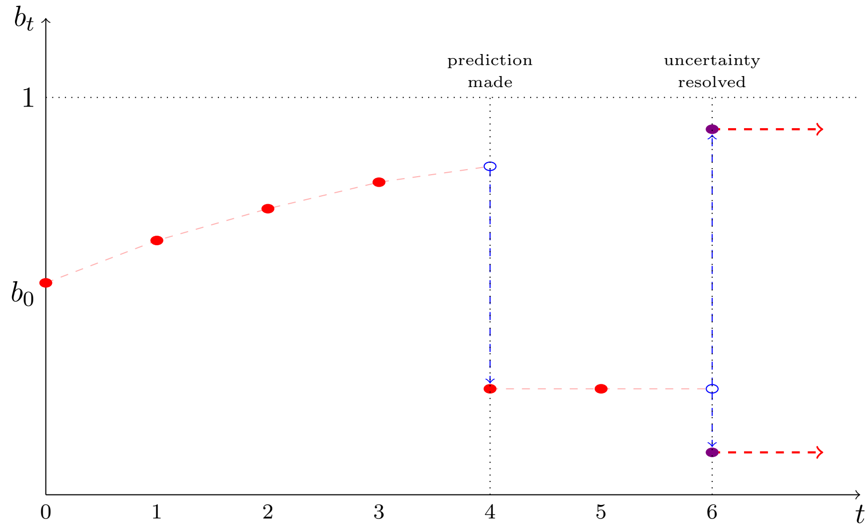

Figure 2: Example forecaster's reputation path.

The paper is organized as follows. Section II contains a review of the relevant literature. Section II presents our results in the simplest setting. In Section II we formulate the general model. The main results are presented in Section II. Section II contains extensions and alternative specifications. Section II concludes. All proofs are relegated to the Appendix.

Relation to the Literature

The current paper mainly contributes to two strands of literature: communication with *career concerns* and the *timing of communication*.

[3]Under special assumptions there also exist degenerate equilibria, in which quack never makes any predictions for the fear of being proved wrong. See Section II for details.

The importance of career concerns for informative communication was first argued by Holmström [1999]. One of Holmström's original examples illustrates that an analyst may be reluctant to truthfully reveal his private information for fear of making a mistake and appearing incompetent, preferring instead to herd with public information or reports of other experts.[4] Other papers have argued that some cohort of analysts – or even all of them in some settings – may, conversely, resort to extreme reports, overstating their private signals in order to separate themselves from "herders" (see Prendergast and Stole [1996], Graham [1999], Hong, Kubik, and Solomon [2000], Lamont [2002], Ottaviani and Sørensen [2006b], Mariano [2012]). Either way, it is generally agreed that analysts' career concerns make information transmission noisy.[5] Dewatripont, Jewitt, and Tirole [1999], Prat [2005], Ottaviani and Sørensen [2006c], and Rodina [2017] give various general characterizations of communication outcomes in the presence of career concerns and their dependence on the information structure of the game.

Of all papers mentioned above only a few look at the dynamics of announcements. In the model of Prendergast and Stole [1996] the expert obtains his private information gradually over time, and his competence determines the speed of learning. They establish that the experts overreact to early pieces of information in order to establish their reputation for competency early on, while as time progresses, they become too reluctant to change their decisions and thus underreact to late information. Predictions of a model by Graham [1999] can be interpreted in a similar way.[6] Hong, Kubik, and Solomon [2000] and Lamont [2002] find a completely opposite pattern in the data: as the experts become older and more established, they usually make more extreme predictions. Li [2007] shows theoretically that when an analyst acquires multiple pieces of information over time, changing one's prediction can act as a signal of competence. However, timing of the prediction or a decision is never a choice variable for the analyst in these papers. Our paper fills this gap by examining how an analyst can manipulate his reputation by strategically choosing the timing of his prediction.

Keskek, Tse, and Tucker [2014] provide evidence from the field that competent experts tend to make their reports earlier – so earlier reports are more informative and are perceived more favorably, – and explain this through preemption mechanisms. We show that competition is *not*

[4] This idea was picked up and greatly extended upon by the literature that followed: see Scharfstein and Stein [1990], Trueman [1994], Ely and Välimäki [2003], Ottaviani and Sørensen [2006a], Dasgupta and Prat [2008].

[5] Zábojnik [2001], Ely and Välimäki [2003] and Klein and Mylovanov [2017] argue that if all analysts are patient enough then this noise vanishes and communication efficiency is restored. Backus and Little [2018] show that making analysts admit uncertainty (not knowing the answer) is also not a trivial problem in the presence of career concerns.

[6] Bernhardt, Wan, and Xiao [2016] observe inertia in financial analysts' predictions, but their explanation of this phenomenon does not rely on career concerns.

necessary for this phenomenon to arise. Bernhardt, Campello, and Kutsoati [2006] also explore competitive prediction markets and discover strong anti-herding dynamics in the data.

The second large (and growing) strand of literature this paper contributes to is that on dynamic communication and, especially, the timing of communication.[7] Guttman, Kremer, and Skrzypacz [2014] provide a notable illustration on the importance of timing in communication. In the context of dynamic disclosure, they show that the same piece of hard (verifiable) information can induce different reactions when disclosed at different times. In other related papers, Guttman [2010], Acharya, DeMarzo, and Kremer [2011], Aghamolla and An [2015], and Gratton, Holden, and Kolotilin [2017] also investigate optimal timing in the context of dynamic disclosure of verifiable information. In contrast to these papers, we deal with soft information, which cannot be credibly disclosed. Grenadier, Malenko, and Malenko [2016] study a setting in which the informed expert uses timing of his (non-verifiable) report to manipulate the timing of the observer's decision. A separate literature explores dynamic revelation of static information and finds, to some surprise, that even if all agents possess all of their respective information in period zero, allowing for multi-period communication may sometimes allow for higher payoffs to some or all parties.[8] All of the aforementioned communication models assume direct conflict of interest between the sender(s) and the receiver(s). Our model of career concerns is different in this regard, since all barriers to truthful communication stem instead from the conflict within the senders' market, namely between competent and incompetent forecasters.[9]

Finally, our paper takes the market for predictions as given rather than designing it in such a way as to extract the most information from the analyst. A general approach to dynamic mechanism design when experts have evolving private information has been proposed by Pavan, Segal, and Toikka [2014]. A sub-field of mechanism design explicitly deals with optimal statistical testing of experts' competence (knowledge of a signal-generating process): see Olszewski [2015] for a recent survey.[10] Our paper is different from this literature in that it does not give the observer the power to design payoffs or information feedback. Instead, it asks the question of whether market forces alone can enable informative communication.

[7]For dynamic models of *repeated* communication, where the sender does not have the choice of timing, see Sobel [1985], Bénabou and Laroque [1992], Morris [2001], Pavesi and Scotti [2017], Alizamir, de Véricourt, and Wang [2018].

[8]See Aumann and Hart [2003], Krishna and Morgan [2004], Alonso and Rantakari [2013], Chen, Goltsman, Hörner, and Pavlov [2017], and Lipnowski and Ravid [2019].

[9]Curiously, effects similar to career concerns models can be obtained in communication settings with sender-receiver conflict where the sender's deceit can be detected with positive probability. For examples of such models see Dziuda and Salas [2019] and Drugov and Troya-Martinez [2019].

[10]The most recent contributions to this field include Di Pei [2016], Ginzburg [2019], Smolin [2018], and Deb, Pai, and Said [2018].

Illustrative Example

This section presents an example that showcases our main results in the simplest setting. Suppose there are two periods $t = 1, 2$ and a binary state $\omega \in \{G, B\}$, which is initially not known to anybody and is publicly revealed in the end of period $T = 2$. Assume that players do not discount the future, and that states are ex ante equally probable, i.e. $\Pr(\omega = G) = \Pr(\omega = B) = \frac{1}{2}$.

There are two players: a forecaster and an observer. The forecaster is, with equal probabilities, either an *expert*, or a *quack*. The forecaster privately knows his type, but the observer does not. The expert has a chance λ_t to privately learn the state in period $t = 1, 2$. With positive probability, the expert also remains unaware of the state, i.e., $\lambda_1 + \lambda_2 < 1$. The quack never receives any private information about the state. In any of the two periods before the state is publicly revealed, the forecaster can send one cheap talk report $m \in \{G, B\}$ to the observer, indicating his prediction about state ω. The report is not verifiable, i.e., the forecaster's private information can not be made observable to the public. At the end of each period the forecaster receives a "reputation payoff" equal to the probability that the observer assigns at that moment to the forecaster being an expert.

We will look for an equilibrium in which the expert is honest: he reports according to his private information as soon as he obtains it and never makes an unfounded prediction or reports contrary to his information (this behavior will be optimal in equilibrium). How would the quack behave in such equilibrium, and how should the market react to either report and to a lack thereof?

There are five actions available to the forecaster in the game: he can report that the state is G or B at $t = 1, 2$ or stay silent throughout. An honest expert plays all five actions with positive probability. It is immediate then that the quack must do the same in equilibrium – if either action is only taken by the expert and never by the quack, then following this path gives the forecaster the highest possible reputation from that point onwards and, therefore, the highest possible continuation payoff. This would strictly dominate any alternative path of play available to the quack at the respective period.

Therefore, the quack must be indifferent, in particular, between reporting that the state is G at $t = 1$ and $t = 2$.[11] Denote by b_t the belief about the forecaster's competence at the end of period t in case no report was made in period t; by $b(m, t)$ the belief after report m at period t was made; and by $b^\omega(m, t)$ the belief after report m at period t was made, and the state turned out to be ω.

[11] We will use message $m = G$ to illustrate our results, but all arguments apply equally to either message.

The indifference condition between the two reports for the quack is then given by

$$b(G,1) + \left[\frac{1}{2} \cdot b^G(G,1) + \frac{1}{2} \cdot b^B(G,1)\right] = b_1 + \left[\frac{1}{2} \cdot b^G(G,2) + \frac{1}{2} \cdot b^B(G,2)\right]. \tag{1}$$

Note that the honest expert is never wrong, since he only makes a report if he knows the state. Therefore, if the forecaster made a prediction which turned out to be incorrect, he is definitely a quack: $b^B(G,1) = b^B(G,2) = 0$.

We have assumed that the expert always reveals his information at $t = 1$ if he has it. However, he does have an option to delay his report until the second period if he already knows the state at $t = 1$. To ensure that there is no delay, the following has to hold:

$$b(G,1) + b^G(G,1) \geqslant b_1 + b^G(G,2). \tag{2}$$

Note that the expert's expected utility only differs from that of the quack in the probability of guessing the state correctly – the expert knows that his private signal is correct. The two expressions (1) and (2) together produce our main results described below.

Early correct reports are rewarded higher ex post. Subtracting equality (1) from (2), we immediately obtain that $b^G(G,1) \geqslant b^G(G,2)$. Early reports must thus be rewarded with higher reputation to incentivize the expert to reveal his information in a timely manner. Note that this only applies to reputation after the state was revealed.

Reporting harms reputation. Combining (1) and (2), we also infer that $b(G,1) \leqslant b_1$, and by analogy we can obtain $b(B,1) \leqslant b_1$. Therefore, any report at $t = 1$ must be worse than not making a report.

Reputation of a silent expert improves. By the martingale property of beliefs, we note that $b(G,1), b(B,1)$, and b_1 must average out to b_0. The inequalities we just obtained then imply that $b_1 \geqslant b_0$: if reporting harms reputation then staying silent must improve it. With slightly more work, one can also show that $b_2 \geqslant b_1$.[12]

Early reports are more precise. The previous observations almost immediately imply that earlier reports contain more information about the state. Indeed, $b^G(G,1) \geqslant b^G(G,2)$ and

[12] By the martingale property, $b(G,2), b(B,2)$, and b_2 must average out to b_1, and we know that $b(m,2) \leqslant b(m,1) \leqslant b_1$ for $m = G, B$. Therefore, $b_2 \geqslant b_1$.

$b^B(G, 1) = b^B(G, 2) = 0$, therefore $b(G, 1) \geqslant b(G, 2)$, since by the martingale property of beliefs, $b^G(m, t)$ and $b^B(m, t)$ should average out (from the observer's perspective) to $b(m, t)$. Because $b_1 \geqslant b_0$ the latter inequality means that the earlier of the two reports is relatively more likely to be made by the expert – which immediately implies that it is more informative than the latter one.

The following sections expand on the analysis of this example in a general framework and show that the insights demonstrated above are quite general.

The Model

Primitives

Time is discrete and finite: $t \in \{0\} \cup \mathcal{T}$ where $\mathcal{T} \equiv \{1, \ldots, T\}$ for some $T > 0$. An underlying standard probability space is implied throughout the paper. The probability measure on this space is denoted by P.

State of the world. There is a binary state of the world ω which can be either *good* or *bad*: $\omega \in \{G, B\}$. The commonly held prior belief that the state is good is $P(\omega = G) = p_0 \in [\frac{1}{2}, 1)$. Initially the state is uncertain; at the end of period T the state is revealed.

Players. There are two players: an observer (she) and a forecaster (he). Both players live for T periods and do not discount the future.

The forecaster has a binary type $\gamma \in \{E, Q\}$: he can be competent or incompetent or, as we call them, an expert (E) or a quack (Q) respectively. The type is privately known by the forecaster, but is not known by the observer. The observer's initial belief that the forecaster is competent is $b_0 \in (0, 1)$.

The observer has no actions in the model.[13] In every period t she updates her beliefs p_t about the state of the world and b_t about the forecaster's competence. It will prove convenient to represent these beliefs as likelihoods rather than probabilities, so let $\rho_t = \frac{p_t}{1-p_t}$ and $\beta_t = \frac{b_t}{1-b_t}$.

At some random time $t^* \sim F(t)$, which is not known to anybody, the competent forecaster observes a signal $\eta^* \in \{G, B\}$ about the state, with precision $\pi := P(\eta^* = G|\omega = G) = P(\eta^* = B|\omega = B)$. For most of the paper we assume $\pi = 1$, but in Section II we show that all results continue to hold in case of imperfect signals, $\pi \in \left(\frac{1}{2}, 1\right)$, given some extra conditions. We assume that $F(t)$ is a measure with full support on $\mathcal{T}$ and that $F(T) < 1$, i.e., there is a positive probability

[13] In the discussion surrounding the model, we assume that she is interested in information about state. To fix ideas, one may think that the observer chooses a binary action from $\{G, B\}$ at time T and receives a fixed reward if and only if her action matches the state – but we do not model this decision explicitly.

that the signal arrives at any time t and it is possible that it never arrives. We denote the conditional probability (hazard rate) of signal arrival in period t as $\lambda(t) := \frac{F(t)-F(t-1)}{1-F(t-1)}$.

The forecaster receives a per-period "reputation payoff" $w(\beta_t)$ which depends on the observer's belief about the forecaster's competence held at the end of period t. We assume $w(\cdot)$ to be strictly increasing in its argument. As a normalization, we let $w(0) = 0$. After the state is revealed, the forecaster receives a terminal payoff $w^c(\beta_T)$, representing the forecaster's continuation value from the reputation he has accumulated. We assume that $w^c(\cdot)$ satisfies the same requirements that we impose on $w(\cdot)$. Payoffs are interpreted as coming from some external source rather than the observer directly. A highly regarded analyst can bargain higher wage from employers in the labor market, while all of the interested public acts as the observer in forming analyst's reputation.

Communication. In any period $t \in \mathcal{T}$ the forecaster can send a report $m \in \{G, B\}$ to the observer, indicating his prediction about state ω. The report is not verifiable, i.e., the forecaster's private information is not ever observable and/or contractible. Additionally, we assume that the forecaster can send at most one report throughout the game.[14]

Timing

At time $t = 0$, the state of the world ω and the forecaster's type γ are realized; forecaster's private signal realization η^* and signal arrival time t^* are drawn from respective distributions. After that, in every period $t \in \{1, ..., T-1\}$ the stage game proceeds as follows:

1. If $t = t^*$ and the forecaster is competent, he observes the realization of η^*;
2. The forecaster updates his belief about the state conditional on observed η^* (if any) and decides whether to send a report $m \in \{G, B\}$ to the observer;
3. The observer updates her beliefs about the state p and about the forecaster's competence b conditional on the forecaster's report or lack of thereof;
4. The forecaster receives payoff $w(\beta_t)$;

In period T steps 1 and 2 take place as above, but instead of steps 3 and 4 the following happens:

3. State ω is publicly revealed;
4. All players update their beliefs accordingly;

[14]This constraint should not be seen as restrictive since the forecaster receives at most one private signal by time T.

5. The forecaster receives a terminal lump-sum payoff $w^c(\beta_T)$.

Histories and State Variables

A *message history* is $\mu_t = (m, s)$ if report m has been made in period $s \leqslant t$ and $\mu_t = \varnothing$ otherwise. A *public history* h_t^p is a tuple consisting of the variables that are publicly observable at the beginning of period t: $h_t^p = (t, \mu_{t-1})$. The forecaster possesses private information about his type and his private signal in addition to whatever is publicly known. We define a type-γ forecaster's *private history* as $h_t^\gamma = (h_t^p, \eta_t^\gamma, t^\gamma)$, where η_t^γ describes forecaster's private information:

- $\eta_t^\gamma = \varnothing$ if no signal was observed in period t or before,
- $\eta_t^\gamma = G$ if signal $\eta^* = G$ was observed in period t or before,
- $\eta_t^\gamma = B$ if signal $\eta^* = B$ was observed in period t or before.

Variable t^γ indicates the arrival time of this information, with $t^\gamma = 0$ meaning no information has yet arrived. For quacks we have that $t^Q = 0$ and $\eta^Q = \varnothing$. For experts these variables can be expanded as $t^E = t^* \cdot \mathbb{I}(t \geqslant t^*)$, and $\eta^E = \eta^*$ if $t \geqslant t^*$, and $\eta^E = \varnothing$ otherwise. Notably, values $(\eta_t^\gamma, t^\gamma)$ are only nontrivial for the expert, thus the quack's private histories are equivalent to public histories, and hereinafter we will treat them as such. We also let $-\eta$ and $-m$ denote the "opposites" of η and m respectively: e.g., if $\eta = G$ then $-\eta = B$.

The Forecaster's Problem

At every history the forecaster decides whether to send a report and, if yes, which report to send. The forecaster's pure strategy is thus a mapping from private histories h_t^γ to the set of feasible messages (which equals $\{\varnothing, G, B\}$ if no report has yet been made and $\{\varnothing\}$ otherwise, since we restrict forecaster to sending at most one message throughout the game). The forecaster's mixed strategy is, as usual, a probability distribution over pure strategies. To simplify the analysis, the following restriction is imposed on strategies:

Assumption 1 (Amnesia)**.** *At any pair of histories $\bar{h}_t^E$, $\bar{\bar{h}}_t^E$ which differ only in signal arrival times $\bar{t}^* < \bar{\bar{t}}^* \leqslant t$, the strategy of the expert must be the same.*

This assumption requires that after the private signal is observed, the expert's reporting strategy does not depend on its arrival time t^*. One may think of this as the expert not remembering when he received the information (but the information itself is never forgotten). This restriction bans strategies like "send a report two periods after receiving a signal". This, however, should not be considered a loss of generality, as the timing of signal arrival is neither observable by anyone except the expert, nor payoff-relevant for any player, so can be seen as nothing more than the expert's private randomization device.

Amnesia together with the fact that only histories with $\mu_{t-1} = \varnothing$ involve non-trivial choice of message allow us to define strategies on the smaller space of tuples (t, η) rather than on all private histories $h_t^\gamma = (t, \mu_{t-1}, \eta_t^\gamma, t^\gamma)$. Therefore, we introduce the forecaster's *behavioral strategy* as $r_\eta^\gamma(m, t)$, which denotes the probability of forecaster γ making report m at time t conditional on having private information $\eta = \eta_t^\gamma$ and having not made any report prior to t.[15] Finally, denote $r^\gamma(m, t) := \mathbb{E}_\eta r_\eta^\gamma(m, t)$. It represents the hazard rate of report (m, t) as perceived by the observer who does not possess the forecaster's private information η (but these objects are still conditional on the forecaster's type).

The forecaster's optimization problem is hence as follows: at every private history h_t^γ such that no report has yet been made ($\mu_{t-1} = \varnothing$), the forecaster of type $\gamma \in \{E, Q\}$ chooses a continuation reporting strategy $\{r_\eta^\gamma(m, s)\}_{s \geqslant t}$ as a solution to the following problem:

$$V_{t,\eta}^\gamma := \max_{r_\eta^\gamma} \mathbb{E}\left[\sum_{s=t}^{T-1} w\left(\beta(h_s^p)\right) + w^c\left(\beta(h_T^p)\right) \middle| t, \eta, \mu_{t-1} = \varnothing\right] \tag{3}$$

subject to evolution of $\beta(h_s^p)$ described in the following subsection. The expectation is taken over all future histories. We also introduce a shorthand notation for the forecaster's continuation value from making report m in period τ at history h_t^γ:

$$W_{t,\eta}^\gamma(m, \tau) := \mathbb{E}\left[\sum_{s=t}^{T-1} w\left(\beta(h_s^p)\right) + w^c\left(\beta(h_T^p)\right) \middle| t, \eta, \mu_\tau = (m, \tau)\right].$$

With this notation we have that report (m, t) is optimal at t if and only if $V_{t,\eta}^\gamma = W_{t,\eta}^\gamma(m, t)$. Moreover, we use $W_{t,\eta}^\gamma(\varnothing)$ to denote the respective value from not making any report until the end of period T (i.e., conditional on $\mu_T = \varnothing$). Finally, as the quack never receives the private signal, we suppress subscript η when talking about $V_{t,\eta}^Q$ and $W_{t,\eta}^Q(m, \tau)$, and refer to these objects as V_t^Q and $W_t^Q(m, \tau)$ respectively.

[15] This is a game of perfect recall, hence by Kuhn's theorem behavioral and mixed strategies are equivalent.

Beliefs

Two important characteristics of any public history h_t^p are public beliefs about the type of the forecaster and about state of the world, $b(h_t^p)$ and $p(h_t^p)$. Recall that $h_t^p = (t, \mu_{t-1})$. We will use this together with the structure of our model to introduce the following labels for beliefs:

$$b(m,t) := b(s,(m,t)) \qquad\qquad p(m,t) := p(s,(m,t))$$

$$b_t := b(t,\varnothing) \qquad\qquad p_t := p(t,\varnothing)$$

for all $s \geqslant t$. In this notation, $b(m,t)$ is the belief about the forecaster's type held by the observer at any time $s \geqslant t$ conditional on report m made at time t, and b_s is the same belief held in the absence of any reports. The same applies for the observer's belief about state, and we will use the same notation for ρ and β, where applicable. This notation is well defined because once a report has been made, both beliefs are frozen in place since no further information can be conveyed from the forecaster to the public.

Finally, we let $b^\omega(m,t)$ denote the belief about the forecaster's type given a terminal history $h_T^p = (T,(m,t))$ and given that the state was revealed to be ω.

Equilibrium Definition

We are looking for Weak Perfect Bayesian Equilibria of the game, which consist of a strategy profile $\{r_\eta^\gamma(m,t)\}$ and a belief profile $(b(h_t^p), p(h_t^p))$ such that:

1. strategies r_η^γ solve (21) given the observer's updating rule for $b(h_t^p)$,

2. all players update their beliefs via Bayes' rule on path.

We further adopt three following refinements (in addition to restriction to amnesiac strategies):

(OP) Off-path Pessimism: off the equilibrium path the beliefs are $p = p_0$ and $b = 0$, with the exception that extreme belief $b = 1$ is not updated;

(ML) Message Labeling: $r_G^E(G,t) \cdot r_B^E(B,t) \geqslant r_G^E(B,t) \cdot r_B^E(G,t)$ for any t;

(SY) Symmetry: $r_G^E(G,t) = r_B^E(B,t)$ and $r_G^E(B,t) = r_B^E(G,t)$ for all t.

Off-path pessimism (OP) only makes it easier to sustain any given strategy profile as equilibrium because it makes deviations extremely unappealing for the forecaster. In particular, if there is some

PBE with some off-equilibrium path beliefs, then the same profile of strategies and on-path beliefs would still constitute a PBE when paired with off-path beliefs prescribed by (OP). The exception in (OP) comes into play only if $\pi = 1$ (i.e., expert's information predicts the state perfectly) and only at histories at which the report was supposedly made by an informed expert for sure, but turned out to be incorrect. The exception says that the forecaster is then still believed to be competent. This behavior of beliefs can be explained as the limiting case of the model as $\pi \to 1$.[16]

Message labeling (ML) requires that report m is more indicative of state $\omega = m$ than the other report. This assumption is without loss, since at any history h_t^p we can assign message labels G and B to the two messages in such a way that (ML) is satisfied.

The only requirement that imposes any actual restrictions is symmetry (SY). It requires that the expert treats states and messages equally – if he has evidence of state G, he sends report G with the same probability that he would have sent report B if he had evidence of state B. This assumption is made for tractability, so that the observer's belief about state stays at a constant level $p = p_0$ as long as no report is made.[17] We have no reasons to believe that the predictions of our model would not hold in asymmetric equilibria.

Equilibrium Analysis

This section strives to characterize the set of all Weak PBE of the game. The main question that is answered in this section is as follows: assuming that in some equilibrium reports are only made at some set of periods $S \subseteq \mathcal{T}$, how do the forecaster's strategies look and how does the informativeness of the reports change across different periods? It turns out that all equilibria have quite a lot of common structure. Proofs of all statements presented in this chapter can be found in the Appendix.

Belief Updating

This section specifies how exactly observer's beliefs b and p evolve given the forecasters' strategy profile $\{r_\eta^\gamma(m,t)\}$.

Conditional on the forecaster not making a report, the observer's beliefs are updated as follows:

$$\beta_t = \beta_{t-1} \cdot \frac{1 - r^E(G,t) - r^E(B,t)}{1 - r^Q(G,t) - r^Q(B,t)}. \tag{4}$$

[16]Section II discusses alternatives to (OP) in case of perfect signals.

[17]Silence is informative about the state only if the expert conceals his private signals, and does so differently conditional on different information. Assumption (SY) explicitly prohibits the latter part of this.

Conditional on no report by the end of period T and realized state ω, the observer's terminal belief is

$$\beta^{\omega}(\varnothing) = \beta_0 \cdot \prod_{t=1}^{T} \left(\frac{1 - \sum_{m \in \{G,B\}} \mathbb{E}[r_{\eta}^{E}(m,t)|\omega]}{1 - \sum_{m \in \{G,B\}} r^{Q}(m,t)} \right), \tag{5}$$

which in a symmetric equilibrium reduces to $\beta^{\omega}(\varnothing) = \beta_T$.

Similarly, employing the Bayes' rule we can derive the observer's belief $\beta(m,t)$ following forecaster's report (m,t), and the observer's terminal belief $\beta^{\omega}(m,t)$ given forecaster's report (m,t) and the realized state ω:

$$\begin{aligned} \beta(m,t) &= \beta_{t-1} \cdot \frac{r^{E}(m,t)}{r^{Q}(m,t)}, \\ \beta^{\omega}(m,t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_{\eta}\left[r_{\eta}^{E}(m,t)|\omega\right]}{r^{Q}(m,t)}. \end{aligned} \tag{6}$$

Finally, another relevant belief is the observer's belief about the current state, p_t. As mentioned before, symmetry implies that in the absence of the report this belief is frozen at its initial level, $p_t = p_0$. Following report (m,t) the belief is updated as:

$$\rho(m,t) = \rho_{t-1} \cdot \frac{(1-b_{t-1}) \cdot r^{Q}(m,t) + b_{t-1} \cdot \mathbb{E}_{\eta}\left[r_{\eta}^{E}(m,t)|\omega = G\right]}{(1-b_{t-1}) \cdot r^{Q}(m,t) + b_{t-1} \cdot \mathbb{E}_{\eta}\left[r_{\eta}^{E}(m,t)|\omega = B\right]} = \rho_{t-1} \cdot \frac{1+\beta^{G}(m,t)}{1+\beta^{B}(m,t)}. \tag{7}$$

Supports of the Reporting Times

Given $\gamma \in \{E,Q\}$ and $m \in \{G,B\}$, define support $\mathcal{S} := \{t_1, t_2, \ldots, t_{|\mathcal{S}|}\} \subseteq \mathcal{T}$ as the set of times t at which any report is made:[18]

$$\mathcal{S} := \{t \in \mathcal{T} \mid r^{\gamma}(m,t) > 0 \text{ for some } \gamma, m\} \tag{8}$$

Proposition 1. *In any equilibrium, any report (m,t) for $m \in \{G,B\}$ is made with positive probability by a quack if and only if it is ever made by an expert: $r^{E}(m,t) > 0$ if and only if $r^{Q}(m,t) > 0$.*

The reasoning behind this proposition is as follows. Suppose that there exists (m,t) such that $r^{Q}(m,t) > 0$ but $r^{E}(m,t) = 0$, i.e., report m at t is only ever made by a quack. Then after report (m,t) the observer infers that the forecaster is surely incompetent. This renders report (m,t) to be a dominated reporting strategy for the forecaster – strictly so if we recall that the belief about the

[18] More generally, the support $\mathcal{S}$ is a subset of public histories h_t^p for which $r_{\eta}^{\gamma}(m,t) > 0$ for some γ, η, m. Since a public history in our model consists of current time t and a messaging history μ_t, and reports can only be made at histories with $\mu_t = \varnothing$, it is without loss to define the support as a set of times.

forecaster's type is a martingale. Therefore, there must exist another continuation strategy at time t – i.e., at the public history $h_t^p = (t, \varnothing)$ – that results in strictly positive reputation for at least one period. No forecaster is willing to play strictly dominated strategies, hence this cannot happen in equilibrium. A similar logic is in play in the opposite case – if $r^E(m,t) > 0$ and $r^Q(m,t) = 0$ for some (m,t) – except then reporting (m,t) is a strictly dominant strategy for any type of the forecaster since it yields the maximal possible reputation starting from t for the rest of the game. Reporting (m,t) is then strictly preferred by the quack to any other alternative, which again gives a contradiction.[19]

Informative Reports and Babbling

If report (m,t) is made in equilibrium, this does not by itself mean that it contains any meaningful information about the state of the world or the type of the forecaster. Following Crawford and Sobel [1982], we refer to uninformative reports as *babbling*.

Definition 1. *We say that report* (m,t) *is* babbling *if*

$$b(m,t) = b_{t-1} \tag{9}$$

$$p(m,t) = p_{t-1} \tag{10}$$

Report (m,t) *is* informative *if it is not babbling.*

Condition (9) implies that the report is uninformative about the forecaster's type, while (10) implies that it contains no information about the state.

It turns out that due to restriction that a forecaster can send at most one report, babbling reports in any equilibrium are organized in a specific structure. This is illustrated by the next proposition.

Proposition 2. *Every equilibrium contains a* Godwin point $\bar{t} := \min\{t \in \mathcal{T} \mid V^E_{t,\varnothing} = V^Q_t\}$ *such that:*

1. *All on-path reports* (m,t) *with* $t > \bar{t}$ *are babbling.*

2. *No on-path reports* (m,t) *with* $t \leqslant \bar{t}$ *are babbling. Moreover:*

[19] One may easily show using the same kind of argument that $r^E(G,t) + r^E(B,t) < 1$ if and only if $r^Q(G,t) + r^Q(B,t) < 1$. I.e., a quack stays silent up until time t with positive probability if and only if so does expert. See the proof of Proposition 1 for details.

- *at every $t < \bar{t}$ the expert does not make a report unless he has received the corresponding signal, i.e., $r^E_\varnothing(m,t) = 0$ and $r^E_\eta(m,t) = 0$ whenever $\eta \neq m$;*
- *at $t = \bar{t}$ the informed expert always reports his signal, i.e., $r^E_\eta(\eta, \bar{t}) = 1$.*

"Godwin's law" states that as a discussion on the Internet continues for long enough, the probability of a comparison involving Nazis or Hitler approaches 1.[20] At that point the informative part of the discussion is usually considered finished, and what follows is just babbling. Along similar lines, Proposition 2 says that in our model all equilibria feature at most two phases: early reports are informative, while the late ones do not contain any relevant information about the state or the type of the forecaster.

To understand Proposition 2 it is enough to note that by Proposition 1, $t \in \mathcal{S}$ only if an expert is willing to report at t. His comparative advantage relative to quack is his ability to acquire private signals. Therefore, the expert is only willing to participate in babbling if he has no option to exploit his [current or possibly future] information by sending an informative report – i.e., if the Godwin point $\bar{t}$ has passed and the discourse has descended into babbling. Conversely, whenever an option to make an informative report now or in the future is present (i.e., $t < \bar{t}$), the expert is not willing to report contrary to his private information or make an unfounded report. The only kind of information distortion that he is willing to partake in is delaying information revelation – but even in this case delaying beyond the Godwin point $\bar{t}$ cannot be worth it.

It is worth noting that $\bar{t}$ does not have to be in the interior of the support, so one of the phases may be absent. In particular, if $\bar{t} < t_1$ then all reports are babbling, while if $\bar{t} = t_{|\mathcal{S}|}$ then no babbling takes place in equilibrium. We shall refer to the latter type of equilibria as *informative.*

Definition 2. *An* informative equilibrium *is an equilibrium where all reports in the support are informative.*

Note that in any informative equilibrium with support $\mathcal{S}$, it must be that $\bar{t} = t_{|\mathcal{S}|}$, since the definition directly implies $\bar{t} \geqslant t_{|\mathcal{S}|}$, and the condition $V^E_{t,\varnothing} = V^Q_t$ is satisfied for $t = t_{|\mathcal{S}|}$.

The next proposition shows that the babbling phase may be safely ignored altogether, and without loss of generality we may consider only informative equilibria.

Proposition 3 (Babbling Irrelevance)**.** *For any equilibrium with support $\mathcal{S}$ and Godwin point $\bar{t}$ there exists an informative equilibrium with the same Godwin point $\bar{t}$ and support $\tilde{\mathcal{S}} = \mathcal{S} \cap \{t \leqslant \bar{t}\}$ such that the two equilibria are:*

[20] See "Meme, Counter-Meme" (Wired).

1. *payoff-equivalent for all players,*

2. *strategy-equivalent on $\tilde{\mathcal{S}}$.*

Propositions 2 and 3 together imply that any equilibrium strategy profile with some Godwin point $\bar{t}$ can be obtained from a respective informative equilibrium with the same Godwin point by allowing for some babbling in $\{\bar{t}+1, ..., T\}$.

Finally, to simplify the statements of our results, we will also focus on *reticent* equilibria, as defined below.

Definition 3. *We call an equilibrium* reticent *if* $r^E_{\varnothing}(G, \bar{t}) = r^E_{\varnothing}(G, \bar{t}) = 0$.

In informative reticent equilibria, we then have that for all $t \in \mathcal{S}$: $r^E_{\varnothing}(m, t) = 0$ for all $m \in \{G, B\}$ and $r^E_{\eta}(m, t) = 0$ for $\eta \neq m$. The expert in such equilibria only makes a prediction m if he has received private signal $\eta = m$. The main remaining question is how the quack responds to such an expert's strategy. The following subsection answers this question in the context of informative reticent equilibria, and in Section II we show how these results extend to equilibria that are not reticent.

Main Results

This section fixes an arbitrary support $\mathcal{S} = \{t_1, t_2, \ldots, t_{|\mathcal{S}|-1}, t_{|\mathcal{S}|} = \bar{t}\} \subseteq \mathcal{T}$ and explores properties of informative reticent equilibria on $\mathcal{S}$ (assuming they exist). Other kinds of equilibria are explored in Section II. For simplicity we also assume throughout the remainder of Section II that the expert's signals are absolutely precise ($\pi = 1$); this assumption is relaxed in Section II.

To start with, it is useful to understand how equilibria look conditional on the support. Propositions 2 and 3 imply that the the expert only reports when he has already received a private signal, except maybe at the last point of the support. Assumption (SY) then implies that if $r^E(G, t) > 0$ then $r^E(B, t) > 0$ and vice versa. Proposition 1 together with the above leads to the fact that in any informative equilibrium for any $t \in \mathcal{S}$: (1) both reports $m = G$ and $m = B$ are made at t in equilibrium, and (2) both types of forecasters make any given report $m \in \{G, B\}$ at t in equilibrium. Alternatively, one may say that $\mathcal{S} = \{t \mid r^E_{\eta}(\eta, t) > 0\}$ for any η, i.e., in any informative equilibrium the support is a set of times at which the expert discloses some of the information he possesses.

To talk about the informativeness of different predictions about the state of the world, we introduce the following measure:

$$i(m,t) := \ln(\rho(m,t)) - \ln(\rho_{t-1}) = \ln\left(1+\beta^G(m,t)\right) - \ln\left(1+\beta^B(m,t)\right).^{21}$$

This measure shows how likely report (m,t) is to be sent in state G as opposed to state B. Positive values reinforce the observer's belief in state $\omega = G$ after hearing this report, while negative values do the same for state $\omega = B$. Higher absolute values of $i(m,t)$ mean that more information is transmitted by message (m,t) to the observer, meaning that belief $\rho(m,t)$ moves further away from ρ_{t-1}.

Presented next is the central result of our paper, which describes the informational content of reports and the informativeness dynamics. All monotonicity statements in this Theorem are understood in the sense of weak monotonicity.

Theorem 1. *Suppose that $|\mathcal{S}| \geqslant 2$ and an informative reticent equilibrium on $\mathcal{S}$ exists. Then in any such equilibrium the following are true for both $m \in \{G, B\}$:*

1. *later reports are less informative about the state: $|i(m,t)|$ is a decreasing function of t on $\mathcal{S}$;*
2. *the reputation of a silent forecaster improves over time: b_t is increasing in t on $\mathcal{S}$ and constant on $\mathcal{T}\backslash\mathcal{S}$;*
3. *making any report decreases reputation as compared to no report: $b(m,t) \leqslant b_t$ for any $t \in \mathcal{S}$.*

Theorem 1 starts by stating that in any *reticent* informative equilibrium with $|\mathcal{S}| \geqslant 2$ reports should become [weakly] noisier over time. This is required to provide incentives for the expert to disclose the information he possesses. To elaborate, Proposition 1 implies that a quack must be indifferent between all reports (m,t) made in equilibrium. At the same time, the only difference between the expert's and the quack's payoffs comes from their respective probabilities of guessing the state correctly with their report. Therefore, conditional on the quack's indifference, the expert with information $\eta \in \{G, B\}$ in period t effectively maximizes the net premium for guessing the state correctly, as given by

$$\Delta w_\eta(m,\tau) := w^c\left(\beta^\eta(m,\tau)\right) - w^c\left(\beta^{-\eta}(m,\tau)\right),$$

[21] Note that since Bayes' rule is linear in log-likelihoods, $|i(m,t)|$ shows exactly the "strength" of the signal contained in (m,t) in terms of it's effect on the posterior $p(m,t)$ relative to the prior p_{t-1}. That said, our results are not specific to the particular functional form of $i(m,t)$ and are compatible with any other measure of distance between $\rho(m,t)$ and ρ_{t-1} which is increasing in $|\rho_{t-1} - \rho(m,t)|$ for any fixed $\rho(m,t)$.

over all reports (m,τ) with $\tau \geqslant t$. From (ML) and (SY) we know that $\varDelta w_\eta(m,\tau)$ is weakly positive for $m = \eta$ and is weakly negative for $m = -\eta$, hence it is enough to consider $m = \eta$. Moreover, Propositions 1 and 2 together imply that in informative equilibria, $t \in \mathcal{S}$ if and only if an informed expert reports at t. This means that for $t \in \mathcal{S}$ we have

$$(\eta,t) = \max_{m,\tau\in\mathcal{S},\tau\geqslant t} \varDelta w_\eta(m,\tau)$$

or, simply speaking, $\varDelta w_\eta(m,t)$ must be a weakly decreasing function of t on $\mathcal{S}$ for $m = \eta$. Note that in case $\pi = 1$, Proposition 2 implies that $\beta^{-\eta}(\eta,t) = 0$ for all $t \in \mathcal{S}\backslash\{\bar{t}\}$, and therefore $\varDelta w^c_\eta(\eta,t) = w\left(\beta^\eta(\eta,t)\right)$. Finally, as $w(\cdot)$ is strictly increasing, its monotonicity is equivalent to monotonicity of $\beta^\eta(\eta,t)$, which in the end directly translates into that of $|i(\eta,t)|$.

The second and the third statements can be shown using the monotonicity of $\beta^m(m,t)$ derived above, but the main intuition behind them comes from the quack's indifference between all reports made in equilibrium. Take some $t_k \in \mathcal{S}$ and suppose that $b(m,t_k) < b_{t_k}$ for both $m \in \{G,B\}$. Then it should be that $b(m,t_{k+1}) < b(m,t_k)$ for both m, since otherwise any report (m,t_k) dominates any report (m,t_{k+1}) – the former grants higher payoff at t_k, higher payoff between t_{k+1} and T, and higher continuation payoff after T. By the martingale property of beliefs, $b(m,t_{k+1})$ and $b_{t_{k+1}}$ should average out to b_{t_k}, so in the end we have that

$$b(m,t_{k+1}) < b(m,t_k) < b_{t_k} < b_{t_{k+1}}$$

whenever $b(m,t_k) < b_{t_k}$. The same argument extends to all $t \in \mathcal{S}$, granting the second and third statements of Theorem 1. This argument does not preclude monotonicity from going the other way if we start from the inequality $b(m,t_k) > b_{t_k}$ – but this case would generate a sequence $b^m(m,t)$ that is *increasing* in $t \in \mathcal{S}$, which is incompatible with the expert's preferences discussed previously. Finally, the argument above implies that penalties for reporting increase over time: if $t_k, t_{k+1}, t_{k+2} \in \mathcal{S}$ then

$$b_{t_{k+1}} - b(m,t_{k+2}) > b_{t_k} - b(m,t_{k+1}).$$

This is exemplified in Figure 3, where the red solid line shows the reputation path of a forecaster who makes a report at $t = 3$, and the blue dashed line shows that for $t = 4$.

Finally, it is worth noting that even the third statement, which is inherently static, requires $|\mathcal{S}| \geqslant 2$. If $|\mathcal{S}| = 1$ (so $\mathcal{S} = \{\bar{t}\}$) then it is no longer true: one may construct an equilibrium with

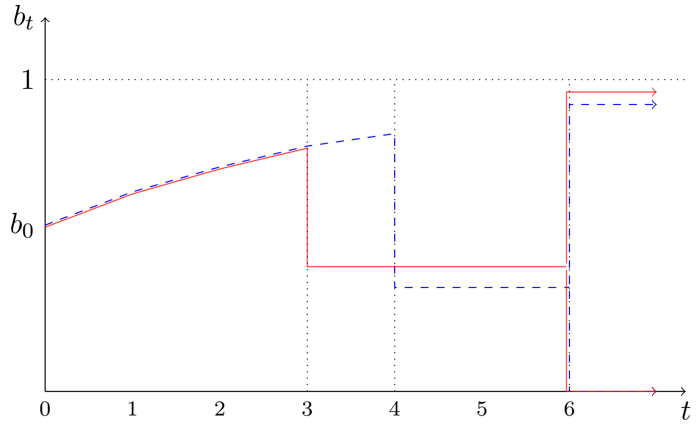

Figure 3: Report penalties increase over time.

$b(m, \bar{t}) > b_{\bar{t}}$ for both $m \in \{G, B\}$. In such equilibrium either report is more likely to be made by an expert than a quack. "Static" equilibria (those with $|\mathcal{S}| = 1$) are in this sense potentially more informative than "dynamic" equilibria, and allowing for reports to be made at more than one point in time may actually be harmful to the informativeness of these reports.

Existence of Informative Equilibria

So far we have discussed properties of equilibria without proving that any equilibria actually exist, but existence of informative equilibria is not a trivial concern.[22] The following Proposition outlines some necessary and sufficient conditions for existence of informative equilibria, which allow to understand some driving forces behind their existence and non-existence.

Proposition 4. *Suppose $w(\cdot)$ and $w^c(\cdot)$ are continuous function. Then*

1. *For any $\bar{t}$ there exists an informative equilibrium with $\mathcal{S} = \{\bar{t}\}$;*

2. *If $w(\beta)$ and $w^c(\beta)$ are convex and $p_0 = \frac{1}{2}$ then a reticent informative equilibrium with arbitrary $\mathcal{S}$ exists;*

3. *If $w(\beta) = \beta^\alpha$ and $w^c(\beta) = \theta\beta^\alpha$ with $\theta > 0$ and $\alpha < 1$, then no informative equilibrium with $|\mathcal{S}| \geqslant 3$ exists.*

Part 1 of Proposition 4 states that at least some informative equilibria always exist. In particular, there always exist equilibria with singleton support, whatever the single period in the support is. At this period experts reveal all private information they have accumulated by then,

[22]Babbling equilibria, on the other hand, always trivially exist.

and any forecaster without private information is also free to make a report in the hopes of guessing the state correctly.

However, the main focus of this paper is on the dynamics of announcements, so we are particularly interested in equilibria with $|\mathcal{S}| > 1$. Part 2 of Proposition 4 gives a sufficient condition for their existence, which is convexity of the payoff function $w(\cdot)$ and symmetry of the two states, $p_0 = \frac{1}{2}$. By the continuity of payoffs, the condition on p_0 can be relaxed to some extent. All else equal, for any convex $w^c(\cdot)$ there exists $\varepsilon > 0$ such that an informative equilibrium for arbitrary support $\mathcal{S}$ exists whenever $p_0 \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$.[23]

Necessary conditions, on the other hand, are not easily obtainable in our model. The reason lies in the fact that the payoff functions $w(\cdot)$ and $w^c(\cdot)$ are only invoked for a finite number of arguments β in any given equilibrium. In particular, given some payoff function $w(\cdot)$ and some equilibrium of the game, we can change values that $w(\cdot)$ and/or $w^c(\cdot)$ take almost everywhere without affecting the equilibrium. This makes necessary conditions difficult to formulate without restricting payoff functions to a specific class, which is what the last part of Proposition 4 does. It states that for at least some class of concave payoff functions the existence of equilibria with large supports ($|\mathcal{S}| \geqslant 3$) completely breaks down.[24] Parts 1 and 3 together illustrate that the main hurdles to existence are tied to intertemporal choice: if the forecaster has no choice of *when* to make a report then existence is certain, while allowing predictions to be made at multiple points in time may in some settings lead to complete breakdown of communication.

The reason for non-existence is connected to the expert's dynamic incentive compatibility constraints. This is because $t \in \mathcal{S}$ if and only if the informed expert makes a report at t, so he should be willing to do so instead of delaying his report until a later date. This leads to phenomena described in parts 2 and 3 of Theorem 1. In particular, any report has to reduce the forecaster's reputation, so the only reason to make the report for the quack is a gamble for the terminal reputation: he should be willing enough to make a guess, understanding that it may be incorrect. A certain degree of risk-loving on behalf of the forecaster is required for such a strategy profile to constitute an equilibrium. Conversely, if the quack is too risk-averse then strategy profiles with $|\mathcal{S}| \geqslant 3$ cannot satisfy the incentives of both forecaster types at the same time.[25] The formal

[23]An empirical inquiry by Bernhardt, Campello, and Kutsoati [2004] also proposed the convexity of payoffs in reputation as an explanation of the observed dynamics of the analysts' reports. In their case, the observed phenomenon was strong anti-herding in predictions.

[24]The jump from $|\mathcal{S}| = 1$ to $|\mathcal{S}| \geqslant 3$ is tied to the special features of the Godwin point, which precludes us from making sharp statements about equilibria with $|\mathcal{S}| = 2$. See also Section II.

[25]Remember that $w(\beta_t)$ is a function of $\beta_t = \frac{b_t}{1-b_t}$ which itself is a convex function of b_t. Therefore, even with $\alpha = 1$ the forecaster is still risk-loving, so all talks of risk-loving and risk-aversion are in the relative sense (one may easily verify that coefficients of both absolute and relative risk-aversion are monotone in α).

argument is somewhat more subtle and can be found in the Appendix.

The intuition above naturally leads to the question: do there exist equilibria, given enough risk-aversion on forecaster's behalf, in which the quack is too reluctant to make his report for fear of guessing it wrong? Such equilibria do not require sustaining quack's indifference between all reports, so they should seemingly exist under a wider range of parameters and functional forms. In the current setting the existence of such equilibria violates Proposition 1 and is therefore impossible. In Section II we show that after adopting an alternative assumption on off-path beliefs such equilibria *can* in fact exist, but only if $\pi = 1$.

Comparison of Equilibria

In this section we study how the informativeness of the reports depends on the shape of equilibrium. Simply speaking, we are trying to answer the question of which equilibria are more informative.

We have two characteristics that describe how informative a given equilibrium is: its support $\mathcal{S}$ and two functions $i(m,t)$ for $m \in \{G, B\}$.[26] Their exact meaning, however, is worth clarifying. The informativeness measure $|i(m,t)|$ is effectively a signal-to-noise ratio: it shows how noisy a given message is, *conditional* on the event that this message is sent. The probability of the latter, however, is governed by $\mathcal{S}$, so $|i(m,t)|$ alone does not allow to conclude ex ante how much information will be conveyed at t. Sparser support $\mathcal{S}$ means that reports arrive more rarely in equilibrium and it may take longer for a given piece of information to be disclosed, but it does not necessarily imply that less information is transmitted (as long as $\bar{t}$ is unaffected). To elaborate, any piece of information that is observed by the expert at $t' \notin \mathcal{S}$ is not lost to the void – its revelation is delayed until $t'' = \min\{t \in \mathcal{S} | t > t'\}$ but it *is* reported eventually.

Proposition 5 below summarizes our knowledge of how different equilibria of the game compare to each other in terms of informativeness, given some fixed underlying fundamentals.

Proposition 5. *Assume that two reticent informative equilibria exist with respective supports* $\mathcal{S} = \{t_1, ..., t_k\}$ *and* $\tilde{\mathcal{S}} = \{t_1, ..., t_k, t_{k+1}, ..., t_{k+n}\}$*, and informativeness measures* $i(m,t)$ *and* $\tilde{i}(m,t)$*. Then* $|i(m,t)| \leqslant |\tilde{i}(m,t)|$ *for* $m \in \{G, B\}$ *and all* $t \in \tilde{\mathcal{S}}$*.*

The proposition says that expanding support to the right increases the informativeness of all reports as long as the expert makes no uninformed reports. In other words, this says that "extending

[26] This discussion implicitly focuses on the observer's welfare. forecaster's type is not of interest to the observer, hence $g(m,t)$ is not a variable of interest for us.

the deadline" for reports – in the sense of switching to an equilibrium with larger $\mathcal{S}$ – is always good for the observer. It both allows more information to be transmitted by the informed expert (in case he observes his private information between t_k and t_{k+n}) *and* decreases noise of *all* informative reports (weakly for all $t \leqslant t_k$ and strictly for all $t > t_k$). The intuition behind the latter phenomenon follows from Theorem 1. Simply speaking, the more reporting options are available to quack in a given equilibrium, the thinner he spreads over them. A more detailed argument follows.

Ceteris paribus, extending the support to the right (i.e., adding later dates) implies that the reputation b_t of the silent forecaster should improve at the new dates. This makes an option of staying silent (or making a report at the last point) more attractive to the quack and does not affect his payoff from making a report. By Proposition 1, the quack should be indifferent between these options, so to restore this indifference after expanding the support we have to make reports more appealing to him – which is achieved by prescribing point-wise lower $r^Q(m,t)$ in equilibrium, thereby improving $b(m,t)$ and $b^\omega(m,t)$ and at the same time depressing b_t.

Discussion and Extensions

Delay Equilibria

Although Proposition 2 states that the expert only reports at $t < \bar{t}$ if he has already received a signal, it is still possible that he may delay his report, making it some time *after* he has received a signal (but no later than $\bar{t}$). If this happens, we call an equilibrium a *delay equilibrium.* Conversely, if the expert always discloses his information immediately then we call it a *relay equilibrium.*

Definition 4. *We call an informative equilibrium:*

1. *a* relay equilibrium *if* $r^E_\eta(\eta,t) = 1$ *for* $\eta \in \{G,B\}$ *and for all* $t \in \mathcal{S}$*;*

2. *a* delay equilibrium *otherwise.*

Delay equilibria are very special in two respects. Firstly, unlike relay equilibria, they only exist under knife-edge conditions on parameters. In other words, a generic informative equilibrium is a relay equilibrium, in which the expert discloses his signals immediately. Secondly, delay equilibria necessarily possess more concrete properties than relay equilibria. In particular, Proposition 6 describes how the equilibrium properties described in Theorem 1 specialize in case of delay equilibria.

Proposition 6. *Suppose that* $|\mathcal{S}| \geqslant 2$ *and a reticent delay equilibrium on* $\mathcal{S}$ *exists. Then in any such equilibrium the following are true for both* $m \in \{G,B\}$*:*

1. *the report informativeness $|i(m,t)|$ is constant for all $t \in \mathcal{S}$;*

2. *the silent forecaster's reputation is independent of time: b_t is constant on $\mathcal{T}$;*

3. *the forecaster's reputation is not immediately affected by his report: $b(m,t) = b_t$ for any $t \in \mathcal{S}$.*

Both observations above (that existence conditions and equilibrium properties of delay equilibria present a special case of those for relay equilibria) stem from a common source. In comparison to relay equilibria, delay equilibria impose an extra set of restrictions on players' payoffs: the informed expert must be indifferent between revealing his signal today and delaying his report until the next $t \in \mathcal{S}$. Given that this should be satisfied for both kinds of private signals together with quack's indifference, the set of compatible equilibrium belief profiles shrinks significantly which allows us to provide a significantly stronger version of Theorem 1 for delay equilibria.

Informative Equilibria without Reticence

When presenting the main result of the paper in Theorem 1 and in Proposition 6, we restricted the set of possible equilibria to *reticent* equilibria. The main result, however, holds without this assumption so long as we exclude $\bar{t}$ from parts 2 and 3 of the statement.

Proposition 7. *Suppose that $|\mathcal{S}| \geqslant 3$ and an informative equilibrium on $\mathcal{S}$ exists. Then in any such equilibrium the following are true for both $m \in \{G, B\}$:*

1. *later reports are less informative about the state: $|i(m,t)|$ is a decreasing function of t on $\mathcal{S}$;*

2. *the reputation of a silent forecaster improves over time: b_t is increasing in t on $\mathcal{S}\backslash\{\bar{t}\}$ and constant on $\mathcal{T}\backslash\mathcal{S}$;*

3. *making any report decreases reputation as compared to no report: $b(m,t) \leqslant b_t$ for any $t \in \mathcal{S}\backslash\{\bar{t}\}$.*

Proposition 7 differs from Theorem 1 and Proposition 6 in two respects: it requires $|\mathcal{S}| \geqslant 3$ and excludes $\bar{t}$ from statements 2 and 3. The common reason behind both of these changes is that the Godwin point $\bar{t}$ differs from other points in $\mathcal{S}$. Its distinctive feature is allowing $r^E_\varnothing(m,\bar{t}) > 0$ – that an uninformed expert makes a report, – while from Proposition 2 we know that $r^E_\varnothing(m,t) = 0$ for all $t < \bar{t}$. This can generate situations in which statements 2 and 3 are no longer true at $\bar{t}$, i.e., some report m may have $b(m,\bar{t}) > b_t$, while silence would decrease b_t. This, however, does not affect the first part of the proposition: the reports made by the uninformed expert at $\bar{t}$ are uninformative, and thus only add more noise, amplifying the effect of decreasing informativeness as compared to reticent equilibria.

Ideal Equilibria

Informative equilibria with nontrivial supports need not exist with non-convex payoffs, as evidenced by Proposition 4. A question arises: are babbling and small-support equilibria the only possible outcomes when forecasters are too risk-averse? The answer is "not necessarily".

The key to answering this question is assumption (OP). It requires that once a forecaster has gained perfect reputation it persists forever – even if a forecaster's prediction turned out to be wrong when it could not happen in equilibrium (which is the case if the expert is supposed to report in equilibrium only if he has the respective signal). This is a limiting case of the model as $\pi \to 1$, i.e., it can be supported by a perturbation of the model in which the expert's signal is incorrect with vanishing probability – and thus so are his predictions (see Section II for a more extensive discussion of this setting).

However, this is not the only possible perturbation of the model in case $\pi = 1$. One may alternatively think of a version of the model with the infinitesimal number of "crazy" forecasters who are not strategic in their reports and just voice their opinions at random times. Since their number is infinitesimal, Bayes' rule still prescribes that $b(m,t) = 1$ for any (m,t) such that $r^E_\eta(m,t) > 0 = r^Q(m,t) = r^E_\varnothing(m,t)$ with $\eta = m$. However, since an informed expert is never wrong, if such prediction (m,t) turns out incorrect, this would imply that it was actually made by one of the few crazy forecasters who may be competent or not. This could lead to any belief $b^{-m}(m,t) \in [0,1]$.

In this section we substitute (OP) by an alternative assumption (OP') which prescribes the worst possible off-path belief after an incorrect prediction supposedly made by an expert, same as any other off-path history:

(OP) off the equilibrium path the beliefs are $p = p_0$ and $b = 0$, *with the exception that the extreme belief* $b = 1$ *is not updated*;

(OP') off the equilibrium path the beliefs are $p = p_0$ and $b = 0$.

The alternative assumption (OP') allows for the existence of *ideal equilibria*:

Definition 5. Ideal equilibria *are characterized by* $r^Q(m,t) = r^E_\varnothing(m,t) = 0$ *for all* (m,t)*,* $r^E_\eta(m,t) = 0$ *for* $\eta \neq m$*, and* $r^E_\eta(m,t) > 0$ *for some* (m,t) *with* $m = \eta$*.*

Simply speaking, in ideal equilibria the only reports that are ever made are those by informed experts; quacks never voice their opinion. This type of equilibrium is enforced by the worst possible terminal reputation if the forecaster's report turned out incorrect. For this threat to enforce such an equilibrium, the quacks should be afraid of bad reputation more than they should love good reputation in the short term. In other words, the payoff from reputation $w(\cdot)$ must be relatively concave. While we cannot state the necessity of concavity (see Section II for discussion of necessary conditions), we can show the converse: if $w(\cdot)$ or $w^c(\cdot)$ is convex then ideal equilibria do not exist.

Proposition 8. *Under (OP'), if $w(\cdot)$ or $w^c(\cdot)$ is convex then no ideal equilibria exist.*

This is the exact opposite of part 2 of Proposition 4, meaning that ideal equilibria are, informally speaking, complementary to informative equilibria in the sense of existence. On the formal side, the proof of Proposition 8 contains the necessary and sufficient condition for existence of an ideal equilibrium with support $\mathcal{S}$, but this condition is not particularly insightful, and for that reason we do not state it here.

Imperfect Private Signals

In this section we relax the assumption that the expert's signals are perfectly informative about the state and explore the case $\pi < 1$. Note that there is nothing in the intuition behind Theorem 1 implying that $\pi = 1$ is a necessary condition. As long as $\pi > \frac{1}{2}$, the expert's signal is somewhat informative about the state, so his informed report about the state is more likely to be supported by the ex post evidence than the quack's random guess. Therefore, the results should continue to hold.

The proofs of Propositions 1, 2, and 3 continue to hold in case $\pi < 1$ with no further modifications. Proposition 9 below shows that the remaining results continue to hold as well if $w^c(\cdot)$ is either convex, or at least not too globally concave, and the private signal is sufficiently precise.[27]

Proposition 9. *Theorem 1, Propositions 4–7 are true for $\pi < 1$ if either of the following holds:*

1. *$w^c(\cdot)$ is convex;*
2. *$w^c(\cdot)$ is continuously differentiable and there exist $0 < \underline{d} \leqslant \bar{d} < +\infty$ such that $\frac{dw^c(\beta)}{d\beta} \in [\underline{d}, \bar{d}]$ and $\pi > \frac{\bar{d}}{\underline{d}+\bar{d}}$.*

[27]The exception is Proposition 8, since assumption (OP') is equivalent to (OP) when $\pi < 1$.

When describing the intuition behind Theorem 1, we have mentioned that in order to provide incentives for the informed expert to reveal his private information immediately instead of waiting for a later date, the premium $\Delta w_\eta(m,t)$ for guessing the state correctly should be a decreasing function of t on $\mathcal{S}$. An important part of the proof of Theorem 1 consists of showing that decreasing $\Delta w_\eta(m,t)$ is equivalent to decreasing $b^m(m,t)$. The three statements of Theorem 1 then follow almost directly from the latter statement (using the Bayes' rule and the martingale property of beliefs).

The equivalence relation above is simple when $\pi = 1$, since then $b^{-m}(m,t) = 0$, and $w^c(\cdot)$ is a strictly increasing function. Proposition 9 provides two alternative conditions under which the equivalence holds in case $\pi < 1$. If $w^c(\cdot)$ is convex it holds because $b^m(m,t)$ and $b^{-m}(m,t)$ are scalar multiples of each other.[28] The second condition relaxes convexity to just bounded derivative of $w^c(\cdot)$ but the idea is the same: if $\frac{dw^c(\beta)}{d\beta}$ is bounded so that $w^c(\cdot)$ is not too concave globally, and the signal is precise enough, we can establish the connection between $\Delta w_\eta(m,t)$ and $b^m(m,t)$.

It is also worth noting that ideal equilibria outlined in Section II can no longer exist if $\pi \in \left(\frac{1}{2}, 1\right)$. This is because the forecaster who is believed competent with probability one can no longer be punished after his prediction was revealed to be wrong – he can credibly claim that the mistake was made because of an incorrect private signal, rather than due to low competence.

Commitment

Suppose now that the forecaster can commit to a reporting strategy at $t = 0$ after learning his type but before receiving any private information. The forecaster's strategy is not publicly observable. This modification relates our problem to the literature on Bayesian Persuasion and information design, since the forecaster now designs the disclosure strategy subject to the constraints on the information available to him.[29]

The literature on Bayesian Persuasion has demonstrated that commitment power often allows the sender to strictly improve his payoff whenever the optimal communication mechanism is informative.[30] In contrast, it is easy to see that in our setting all forecaster's strategy profiles that were optimal in the absence of commitment remain optimal even if he has commitment power. In particular, the quack is indifferent between all reports sent in equilibrium and not reporting, if

[28]This follows from the observer's belief p_t regarding state being constant in the absence of reports and the rate of arrival of expert's private signal being the same in both states. Due to these assumptions, ratio of $b^m(m,t)$ to $b^{-m}(m,t)$ equals the relative probability of expert having correct versus incorrect information about the state.

[29]The seminal contribution is Kamenica and Gentzkow [2011]; see Bergemann and Morris [2019] for a recent survey.

[30]See Lipnowski, Ravid, and Shishkin [2018].

allowed in equilibrium. This means that his payoff from any action played in equilibrium is the same, and (OP) implies that playing off-path actions is no better. Therefore, conditional on the expert's strategy, the quack cannot improve by committing to a different strategy. On the other hand, the expert's strategy is also optimal given the quack's indifference: given information η, it is optimal for him to send the report (m,t) that maximizes $\Delta w_\eta(m,t)$, and given no information it is optimal to wait for information (until at least $\bar{t}$). Therefore, the forecaster's commitment power does not affect the equilibria identified above.

Conclusion

The paper presents a model of dynamic cheap talk in the presence of career concerns. We discover that the competition between competent (experts) and incompetent (quacks) forecasters imposes plenty of structure on equilibrium outcomes. In particular, we show that to incentivize the experts – whose reports drive the whole market, – to make early predictions, it must be that early reports are perceived more favorably by the public than later reports. Perhaps more surprisingly, we discover that the presence of quacks in the market together with the monotonicity above generates an automatic penalty for any report: a forecaster who makes a prediction will see his reputation plummeting, and he will only be redeemed if his prediction will turn out to be correct. This does not discourage quacks from speaking up, but disciplines their incentives. Moreover, this reputation dynamics implies that for non-trivial equilibria to exist, forecasters' payoffs must be sufficiently convex in reputation, which is the case if, e.g., the premium for being the top forecaster in the field is very large.

These predictions are novel in the literature, and are driven by us explicitly modeling the dynamic payoff structure of the forecasters. Our model accounts for both flow payoffs while the public is still uncertain about the correctness of the forecaster's prediction, and terminal payoffs realized after the true state is revealed.

The model can be extended in multiple directions, e.g., to account for competition among forecasters, or for arrival of public signal in the background. Richer private news processes for forecasters can also add another strategic layer to the timing decision of the forecaster's prediction. All of these are prospective avenues for future research.

Bad News Turned Good: Reversal Under Censorship[31]

joint with Egor Starkov

Introduction

Word of mouth has long been a significant source of information about product features and quality. One of its manifestations in the digital age is online product reviews. Opinions of fellow consumers often seem more trustworthy than sellers' product descriptions, and the sheer numbers of reviews offer a great diversity of viewpoints. However, sellers can undermine this learning channel, and one instrument they often have for doing so is censorship, i.e., removing unfavorable reviews of their own product.[32]

It is reasonable to expect that whenever censorship is possible and its cost is low, it will be employed to at least some extent. A naive conjecture would be that if the seller can censor at will, then no meaningful bad reviews ever remain, and those that do convey absolutely no information. This is because the seller would delete any review that harms sales. However, in practice we observe plenty of informative bad reviews even when costless censorship opportunities exist. This paper asks why sellers may be willing to *not* censor unfavorable reviews.

We demonstrate that disclosing adverse information can be justified in a *dynamic* disclosure model with *heterogeneous audience*. In particular, we build a model of the market where two groups of consumers are present: *naive* consumers are not aware of the seller's ability to censor, while the *sophisticated* consumers are fully aware of it and make proper inferences from the lack of reviews, among other events.[33] In this model a long-lived seller offers a good of privately known quality to a sequence of short-lived consumers. Consumption generates information about the product quality, and this information is relayed to future consumers through product reviews, which may be deleted by the seller.

The main result of the paper (Theorem 3) states that if some – but not too many – consumers in the market are naive, then there exist equilibria in which bad reviews are revealed in a payoff-relevant way. This result is surprising in that it requires that both groups of consumers are present

[31] This paper should be cited as A. Smirnov, E. Starkov. Bad News Turned Good: Reversal Under Censorship. mimeo, 2020. The paper has been accepted to American Economic Journal: Microeconomics.

[32] This is plausible when we are talking about the seller's own online store, where he has absolute power over the content posted on the website – including product reviews. However, censorship is possible in other settings as well, see Section II for the discussion.

[33] See Section II for references to literature providing empirical evidence of consumers' naivetè when making inferences from product reviews.

in the market for bad reviews to be worth revealing. Indeed, if all consumers are naive, then any bad review decreases their belief in product quality, while the absence of reviews makes them think that no one is buying the product – an event which does not by itself contain information about product quality. In this situation, deleting bad reviews is trivially optimal for the seller. On the other hand, if all consumers are sophisticated, then even though they know that the lack of reviews is likely explained by censorship, they still give the seller a benefit of doubt and allow for the alternative explanation (no one having bought the product). It is then still optimal for the seller to delete all bad reviews.

With both naive and sophisticated consumers in the market a richer dynamic arises. What we show is that presence of naive consumers affects the inference that sophisticated consumers make from observing a bad review. In particular, the seller can now signal the quality of his product to sophisticated consumers by revealing bad reviews. This is costly for the seller, since it decreases sales to naive consumers. The cost, however, is smaller for the seller with a high-quality product. This is because he expects more good reviews in the future, and it is thus easier for him to regain the reputation among naive consumers. The dynamic aspect, therefore, also plays an important role in our results.

We show that signaling is the only motive to reveal bad reviews in our model: Theorem 2 states that a bad review is revealed by the seller *only* if it will increase his reputation among sophisticated consumers. The meaning of bad reviews is thus reversed for sophisticated consumers. On the surface, the necessity of such reversal for bad reviews to be worth revealing can be explained by a very simple argument: bad reviews harm the seller's sales to naive consumers, hence to be worth revealing they must improve sales to sophisticated consumers. However, we show that the link between sales and reputation is much more subtle than this argument makes it seem. This subtlety, in particular, leads to the result that bad reviews must improve the seller's reputation among sophisticated consumers even if they do not directly harm sales to naive consumers.

The reversal is achieved by the dependence of the seller's censorship strategy on the quality of his product. We show that the seller with a high-quality product will never censor bad reviews, while the one with a low-quality product will use a mixed strategy, which assigns positive probabilities to both deleting and revealing every bad review. Therefore, where the naive consumers take bad reviews at face value and perceive them as negative information about the product quality, the sophisticated consumers also extract a positive signal from the fact that this bad review was revealed by the seller, and this positive signal outweighs the inherent negativity of the review.

Taking a step back, the main focus of this paper is on product reviews – and for sake of clarity we will stick to this interpretation throughout – but the model translates naturally to other settings that feature censorship or dynamic disclosure of verifiable information. For example, instead of a seller censoring bad reviews, one may think about a political incumbent censoring news stories in media in an attempt to retain citizens' support. In the context of venture financing, a startup may choose whether to disclose temporary setbacks to the investors or not. A bank may disclose or withhold information about its temporary liquidity deficit in an attempt to prevent a bank run. Our paper implies that in all of these settings, it may be beneficial for the sender to disclose bad news or failures if some receivers are naive, since the rational receivers would take the mere fact of disclosure as a positive signal.

Clearly, even in the context of product reviews, censorship is not the only way the seller can manipulate the information available to the consumer. Posting fake reviews, be it fake positive reviews of own product or fake negative reviews of competing products, is another activity the seller can engage in.[34] While we mostly focus on censorship in this paper, Section II shows that our result continues to hold in the presence of both censorship and fake reviews.

The remainder of the paper is organized as follows. Section II discusses the plausibility of censorship in product reviews and reviews the relevant literature. Section II presents a short example to convey the main idea of the paper. In Section II we formulate the full model. The main results are presented in Section II. Section II contains some further discussion of the model and its extensions, while Section II concludes. All proofs are relegated to the Appendix.

Background and Literature Review

Censorship

The main setting considered in the paper is that of a platform that the seller owns or has moderation rights in. Examples include seller's own website, forum, or Facebook page. In all of these cases the seller is able to remove bad reviews directly. Such deeds are by definition difficult to document, but some claims may be found.[35]

However, it is important to note that the seller does not need to have direct power to remove

[34] About 16% of Yelp reviews are marked as potentially fake (Luca and Zervas [2016]).

[35] The links below lead to articles accusing corrupt employees of Amazon, Reddit, and AirBnB respectively of acting on behalf of companies or sellers: `https://arstechnica.com/information-technology/2018/09/amazon-looking-into-claims-that-employees-delete-bad-reviews-for-cash/`, `http://www.playstationlifestyle.net/2015/11/12/star-wars-battlefront-reddit-mods-bribed/`, `https://qz.com/1333242/airbnb-reviews/`.

bad reviews – he merely needs to convince whoever has this power. For example, some platforms (such as Etsy) allow sellers to try to address buyers' dissatisfaction and ask buyers to remove their negative review if all issues were resolved. While most review aggregators (such as Amazon, Yelp or TripAdvisor) do not allow the sellers to directly remove reviews, convincing, bribing, or harassing consumers into deleting their own reviews are all viable options in those cases. Promising free items or politely asking to contact the company before writing a bad review both have a chance of succeeding at making the consumer remove or alter their bad review, or even not write one in the first place. One extreme method of consumer harassment is SLAPP – Strategic Lawsuit Against Public Participation, – when a seller sues a reviewer primarily to deter other critics from writing negative reviews. While these suits are rarely won in court, they are likely to succeed at forcing the person to delete their review before the suit even reaches court, and/or at intimidating other potential reviewers.[36]

Finally, in some settings the seller may get to choose more favorable reviewers – e.g., a movie distributor picks the critics that get to write the pre-release reviews. In this setting the seller may also ensure that no bad review gets through – either by screening the reviews directly, or by choosing ex ante more favorable reviewers, or through repeated interaction mechanisms.[37]

Academic literature on manipulations in product reviews has focused on the issue of fake reviews (in part because those are easier to observe in the data than deleted reviews which are, by definition, missing from the sample). We are not aware of any papers that deal with censorship in product reviews explicitly, apart from Hauser [2020] who models censorship as depressing the rate of reviews arrival (i.e., censorship is indiscriminate in that model). Political censorship, on the other hand, has received a lot more attention in the literature. Sun [2020] explores a model of dynamic censorship similar to ours, but without naive receivers. Besley and Prat [2006] present a model, in which an incumbent may bribe the media to conceal a bad signal about himself, but focus on the effects of media diversity and independence on political outcomes.[38]

[36] Some examples of [unsuccessful] application of this technique are described in the following news articles from ArsTechnica: `https://arstechnica.com/tech-policy/2015/09/jeweler-tries-to-sue-anonymous-woman-who-left-1-star-yelp-review/`, `https://arstechnica.com/tech-policy/2014/05/lawyers-bully-redditor-for-negative-amazon-com-router-review/`, `https://arstechnica.com/gaming/2016/09/valve-bans-developer-steam-lawsuit-customers-bad-reviews-2/`. One case of such lawsuit being won is described by CBC: `https://www.cbc.ca/news/canada/british-columbia/chinese-wedding-weibo-defamation-1.4556433`. It is by definition more difficult to find documented instances when such harassment was successful – not in the sense of suit being won but in the sense of it forcing consumers to remove their bad reviews.

[37] A recent case here is Disney banning LA Times from pre-release screenings of its movie in retaliation for other recent articles. More details in an ArsTechnica article: `https://arstechnica.com/gaming/2017/11/la-times-you-cant-read-our-thor-review-because-disney-is-mad/`.

[38] Other papers about political censorship include Shadmehr and Bernhardt [2015], Edmond [2013], Egorov, Guriev, and Sonin [2009], Ozerturk [2019], and Chen and Yang [2018], but all of them explore issues that are very different

Disclosure

Our paper belongs to the literature on disclosure of verifiable information, since in our model the seller can only decide whether to disclose any bad reviews written by consumers, but cannot write fake reviews on his own.[39] For a recent survey of the literature on static disclosure see Dranove and Jin [2010].

There is a certain progression that can be observed in this literature.[40] The earliest papers (Grossman [1981]) arrive to the "unraveling" result: if the sender is known to possess a disclosable signal, this signal will be disclosed in equilibrium. "Known to possess information" was later identified as a crucial assumption, without which this result breaks down: the sender has no incentives to disclose adverse information unless it is certain that he has it (see Dye [1985] and Jung and Kwon [1988]).

In the real world very few things are certain, so the question emerged: why can it be profitable to reveal adverse information when it is not certain that the sender has any?[41] Teoh and Hwang [1991], Marinovic and Varas [2016], and Corona and Randhawa [2018] show that bad news may be worth revealing in settings where they may with some probability be discovered by the receiver regardless. Thordal-Le Quement [2014] and Ispano [2018] show that revealing bad news can be used as a signal of the amount of information the sender possesses. Our paper provides a novel motivation for revealing bad reviews, which originates from dynamic incentives in the presence of mixed audience.[42]

In our model, the high-type sender has stronger incentives to reveal adverse information than the low type. This connects us to the literature on countersignaling (see Harbaugh and To, Quigley and Walther [2020], Guo and Shmaya [2019]). That literature, however, derives its results from the presence of an exogenous public signal, which we do not rely on. Heinsalu [2017] obtains a similar kind of signal reversal in the context of dynamic costly signaling (as opposed to disclosure) with no reliance on public signals. The source of this reversal is, however, different from that in our model.

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from those that we focus on.

[39]Although we allow for fake reviews in an extension, see Section II.

[40]We thank the anonymous referee for making this progression salient to us.

[41]E.g., in the context of product reviews, some of the empirical evidence points to the fact that bad reviews are not necessarily harmful and may, in fact, have positive effects on sales or reputation: see Resnick, Zeckhauser, Swanson, and Lockwood [2006], Berger, Sorensen, and Rasmussen [2010] and Maslowska, Malthouse, and Bernritter [2017].

[42]The above only relates to models with fully strategic senders. If one assumes that there exists a behavioral type of the sender which always reveals all information, then the strategic sender would typically have an incentive to mimic it. See Sobel [1985], Kartik and McAfee [2007], Dziuda [2011], and Beyer and Dye [2012] for some examples.

Naivete

Evidence of consumers' naivete when making inferences from product reviews has been provided by Brown, Camerer, and Lovallo [2012] and Li and Hitt [2008]. These papers show that at least some consumers ignore the correlation between other players' actions and their private information. This notion of naivete (which is also implied in our model) has been formalized by Eyster and Rabin [2005], who use the term "cursedness" for this type of irrationality.

More broadly, a wide array of empirical and experimental literature has demonstrated that people play naively even in the very basic disclosure games (see Jin [2005], Deversi, Ispano, and Schwardmann [2020], Jin, Luca, and Martin [2018], and Montero and Sheth [2019]). In particular, Deversi et al. [2020] and Jin et al. [2018] show that people on the receiving side of their disclosure games either play in a way that is very close to a rational player's strategy, or play in way that is very naive. I.e., their players can be separated relatively well into naive and rational receivers. At the same time, when put in the senders' role, all the same people play rationally. These papers demonstrate that naivete is very robust in disclosure setting: even after adopting the role of the sender and going through full strategic reasoning of the sender, people still play naively when switched back to the receiving side.

Crawford [2003] shows that sender's uncertainty about the receiver's sophistication can lead to meaningful communication even in a zero-sum game. We show how these forces manifest themselves in the context of disclosure.

Illustrative Example

This section presents an example that demonstrates our results in the simplest setting. Assume there are three periods $t \in \{1, 2, 3\}$. A long-lived seller offers for sale a product of privately known persistent quality $\theta \in \{H, L\}$ that he has in infinite supply. Price is fixed at $c = 1/3$. Low-quality product always yields utility zero to consumers. High-quality product yields utility 1 with probability $q = 2/3$ and utility 0 with probability $1 - q = 1/3$.

In each period one short-lived consumer arrives at the market with probability $\lambda = 3/4$. Each consumer's prior belief assigns probability $p_1 = 2/3$ to $\theta = H$. The consumer observes past reviews (if any) and updates her belief using Bayes' rule. She then makes a decision on whether to purchase the product. Conditional on the purchase, she consumes the product immediately and leaves a review, which honestly reveals the utility that she received from consuming the product.

Table 1: Three-period equilibrium

Period	Review history	Naive consumers		Sophisticated consumers		L-type seller	H-type seller
		action	belief	action	belief		
1	-	Buy	2/3	Buy	2/3	Mix 50/50	Reveal
2	$\varnothing$	Buy	2/3	Pass	4/9	Delete	Reveal
2	B	Pass	2/5	Buy	4/7	Mix 50/50	Reveal
3	$(\varnothing, \varnothing)$	Buy	2/3	Pass	1/6	-	-
3	$(\varnothing, B)$	Pass	2/5	Buy	1	-	-
3	$(B, \varnothing)$	Pass	2/5	Pass	8/23	-	-
3	(B, B)	Pass	2/11	Pass	8/17	-	-

Note: Seller's strategy in period 3 *is not specified since it is payoff-irrelevant. In review histories,* $\varnothing$ *and* B *mean "no review" and "bad review" respectively. Histories with good reviews are omitted, since after any such history all consumers assign probability one to* $\theta = H$ *and buy the product, and the seller's further censorship strategy is irrelevant.*

We call reviews after utilities one and zero "good" and "bad" respectively. The seller then decides whether to reveal the review. Let r_t^θ denote the [endogenous] probability with which the seller of type θ discloses a bad review conditional on its arrival (and it never makes sense to delete a good review). With probability μ_t the consumer in period t is sophisticated and uses the seller's equilibrium censorship strategy in her inference. With probability $1 - \mu_t$ the consumer at t is naive and assumes the seller never deletes reviews. We let $\mu_2 = 3/4$ and $\mu_3 = 1/2$.

We claim that players' strategies and beliefs as described in Table 1 constitute a Perfect Bayesian Equilibrium of the game. In this equilibrium, the high-type seller reveals all bad reviews, while the low type either deletes the review for sure, or flips a coin, depending on history. To verify that this is indeed an equilibrium, we need to check that:

1. seller's strategy is optimal given consumers' strategies,

2. consumers' strategies are optimal given their beliefs, and

3. consumers' beliefs are consistent with the seller's strategy.

Verifying the correctness of the consumers' beliefs is mechanical and can be easily done by the reader using the Bayes' rule. Consumer strategy optimality is also straightforward given beliefs: in period t she buys the product if and only if her utility from doing so is positive, i.e., $p \cdot q \geqslant c \Leftrightarrow p \geqslant 1/2$. It is, however, instructive to explore the patterns in consumer behavior in this equilibrium, especially since most of the chosen parameter values serve the goal of generating this particular behavior. Period-one consumer buys the product by assumption: $p_1 = 2/3 > 1/2$ (otherwise the market shuts down). After that, any bad review drives the naive consumers away –

until a good review arrives, – since they only perceive it at face value. Sophisticated consumers, on the other hand, continue buying after one bad review because they realize that a high-type seller is more likely to have revealed it. This positive connotation mitigates or even outweighs the negative face value of the bad review.[43] For the same reason, they treat silence (which means that either no consumer arrived, or no sale was made, or the review was deleted) as bad news.

We proceed to argue that the seller's strategy is optimal. As mentioned in the description of Table 1, we only need to look at histories with $t \in \{1, 2\}$ such that the seller has received no good review. This leaves us with three histories: $h_1 = ()$, $h_2 = \varnothing$ and $h_2 = B$. Censorship strategies are only relevant if the seller has received a bad review in the current period, so hereinafter we assume this is the case. At $h_2 = B$ the seller is indifferent regardless of type: neither type of consumer will purchase the product in period three without a good review.

At $h_2 = \varnothing$ the seller faces a more meaningful choice. Deleting a bad review means that naive consumers will continue buying the product: today's purchase came from a naive consumer, and tomorrow only naive consumers will buy as well. Sophisticated consumers will continue to find the lack of reviews disturbing – they will assume that it is due to censorship, hence due to seller knowing his product is bad. On the other hand, revealing a bad review earns the seller credibility in the eyes of the tomorrow's sophisticated consumer, but this comes at the cost of driving away the naive consumer. Revealing a bad review makes the seller lose naive consumers, hence should increase sales to sophisticated consumers. In our example $\mu_3 = 1/2$, hence the seller is exactly indifferent between both options, regardless of own type θ.

Now consider the root history $h_1 = ()$. If the seller has not received a good review, he is bound to lose at least one type of consumers in period two: revealing a bad review loses the naive, having no review loses the sophisticated. In the absence of good news, the two news yield the same expected sales: bad review at $t = 1$ yields a sale w.p. $\mu_2 = 3/4$ at $t = 2$ and none at $t = 3$, while no reviews at $t = 1$ result in sale w.p. $1 - \mu_2 = 1/4$ at $t = 2$ and $1 - \mu_3 = 1/2$ at $t = 3$, for a total of $3/4$. The low-type seller is thus indifferent between the two. The high type, however, expects a positive chance to receive a good review from every sale. Ceteris paribus, it is better to receive this good review sooner rather than later: a good review in period two would generate more sales in period three, while a good review in period three is worthless. In our example, revealing a bad review in period one frontloads the demand to period two, thereby increasing the probability that the seller will have a good review by period three. This logic incentivizes the high-type seller to

[43] Observe also that timing of revelation matters: history $(B, \varnothing)$ is not equivalent to history $(\varnothing, B)$. In this sense, this example also illustrates on a very basic level the idea of Guttman, Kremer, and Skrzypacz [2014].

reveal the bad review at $t = 1$.

The core ideas of the example can be summarized as follows:

- Revealing bad reviews is only worth it if doing so does not decrease total sales, meaning that at least some consumers are expected by the seller to be sophisticated and make strategic (positive) inferences from bad reviews.
- High-type sellers have incentive to front-load demand in order to generate good reviews early. If there are more sophisticated than naive consumers in the market then this entails revealing bad reviews in order to sacrifice naive consumers in an attempt to signal to and attract sophisticated consumers.[44]

The remainder of the paper extends these ideas from the simplified setting of this example to a general environment. Note that generating the right incentives in the example above required varying consumer sophistication over time ($\mu_2 > 1/2 = \mu_3$). This assumption should be seen as an artifact of the example and is not carried over to the main model.

The Model

Time is continuous and infinite, $t \in [0, +\infty)$. A long-lived seller offers for sale a product of privately known persistent quality $\theta \in \{H, L\}$, high or low, that he has in infinite supply. Quality θ is hereinafter referred to as the seller's type. The price of the product is fixed at $c > 0$.[45] Short-lived consumers with a unit demand arrive at the market according to a Poisson process with intensity λ. In other words, the probability that a consumer arrives in any given time interval $[t, t + dt)$ is $\lambda \cdot dt$. All players are assumed to be risk-neutral and evaluate outcomes according to their expected monetary values.

Once a consumer arrives, she and the seller instantaneously play the following stage game, specific elements of which are described in more detail in the following subsections. The consumer who arrives at the market observes all information available to her and decides whether to buy the product. If she does, she receives random utility depending on product quality. After the utility is realized, the consumer leaves a review describing her experience and then leaves the market forever. The seller then decides whether to reveal the review or delete it. If the review is revealed, it is then observed by all future consumers before they make their purchase decisions.

[44] We show in the main model that if there are few sophisticated consumers in the market then full censorship is the only equilibrium.

[45] Allowing the seller to set the price would allow for price signaling, while in this paper we focus solely on signaling through censorship.

We let h_t denote a complete history of the game up to (but not including) time t. It includes current time, the purchase decisions of all consumers who arrived before t, all reviews they wrote, and all respective censorship decisions of the seller. The following subsections elaborate on various parts of the game and introduce notation that will be used throughout.

Consumers

Consumers arrive at the market according to a Poisson process with intensity λ. Each arriving consumer observes the current time and all reviews written by previous consumers that were not deleted by the seller. The consumer does not observe the purchase decisions of the previous consumers and does not observe whether any reviews were deleted.

The consumers' payoffs are as follows. If a consumer leaves the market without buying the product, she receives utility 0. Consuming a high-quality product yields utility 1 with probability q and utility 0 with probability $1-q$, while a low-quality product always yields utility 0.[46] The cost of buying a product is given by its price c.

Each arriving consumer has a "cognitive type" $\gamma \in \{s, n\}$, hereinafter referred to as consumer's type. Sophisticated consumers ($\gamma = s$, share $\mu \in [0, 1)$ of the population) go through full Bayesian reasoning to infer product quality based on published reviews, taking the seller's censorship strategy into account. Naive consumers ($\gamma = n$, share $1 - \mu$) use Bayesian updating for any reviews they observe but are unaware of possible moderation, i.e., they assume that the seller never removes any reviews. For technical reasons, we also assume that naive consumers do not observe or ignore the times at which reviews were written (unlike sophisticated consumers).[47] Consumers' types are i.i.d. within the sequence of arriving consumers.

Let $p^\gamma(h_t)$ denote the probability that consumer of type γ assigns to the product being of high quality given history h_t. She then buys the product at h_t if and only if her expected consumption utility exceeds price c, i.e.,

$$p^\gamma(h_t) \cdot (q \cdot 1 + (1-q) \cdot 0) + (1 - p^\gamma(h_t)) \cdot 0 \geqslant c,$$

or, equivalently, $p^\gamma(h_t) \geqslant \bar{p}$ where $\bar{p} := c/q$. This behavior will be taken for granted for the

[46] High utility is thus a conclusive evidence that the product is of high quality. This assumption of "conclusive good news" is relatively standard in the experimentation literature (see Keller, Rady, and Cripps [2005] and the subsequent literature) since it makes the models a lot more tractable. We relax this assumption in Section II and show that the main result survives under arbitrary information structures. On the other hand, it is crucial for our results that bad news are suggestive rather than conclusive, see Section II.

[47] This assumption can be easily disposed of at the cost of more complicated assumptions about off-equilibrium-path beliefs (described in Section II).

remainder of the paper. To avoid triviality, the parameters are assumed such that $\bar{p} \in (0, 1)$ and the consumers' prior is $p^\gamma(h_0) \geqslant \bar{p}$. We further assume that consumers buy the product when indifferent.

In addition, it will prove convenient to have a separate piece of notation for updated beliefs. Let $f^\gamma(h_t)$ denote the belief p^γ of a consumer arriving in the moment $t + dt$ following history h_t and observing that a bad review was posted at t.

After the utility is realized, the consumer leaves a review. We assume that all consumers leave reviews and do so truthfully: if the consumer received utility 1, she leaves a "good review", while after utility 0 she leaves a "bad review".[48]

The Seller

The seller is long-lived and discounts future at rate $r > 0$. He always observes the complete history h_t of the game so far, as well as his own type θ. We assume that the seller has zero cost of producing the product and thus receives profit c from every purchase. Therefore, instantaneous expected flow profit for type θ seller at history h_t, given that consumers' purchasing decisions are as described above, is equal to $\lambda c \pi(h_t)$ where $\pi(h_t)$ measures expected sales per arriving consumer:

$$\pi(h_t) = (1 - \mu) \cdot \mathbf{1}(p^n(h_t) \geqslant \bar{p}) + \mu \cdot \mathbf{1}(p^s(h_t) \geqslant \bar{p}),$$

with $\mathbf{1}(\cdot)$ being an indicator function. Then the seller's discounted future profit (normalized by $1/\lambda c$) is given by

$$V^\theta(h_t) = \mathbb{E}\left[\int_t^{+\infty} e^{-r(u-t)} \pi(h_u) du \mid h_t, \theta\right], \tag{11}$$

where the expectation is taken over future histories h_u. Note that seller's type θ enters (21) only through this expectation. Also, conditioning on h_t implies that this value function is evaluated *before* the seller knows whether a consumer (and, consequently, a review) arrives at time t.

The seller only has a nontrivial choice of action at those histories at which a consumer arrives and writes a review. The seller then has to decide whether to disclose it or not.[49,50] Any good

[48] All results continue to hold if only some positive share of all consumers leave reviews.

[49] We assume that deleting a review is costless. This may seem to contradict some motivating examples in which deleting reviews is costly, as the company/government has to sustain a customer service/censorship apparatus. However, we argue that *if* the company engages in censorship, the marginal cost of deleting another review is essentially zero.

[50] We assume that reviews cannot be held in a "moderation queue" and revealed later, as well as that published reviews cannot be deleted in the future. The former assumption is made mostly for convenience and has no effect on our results, but the latter assumption is crucial. It can be justified by the folk wisdom that "nothing can be deleted from the Internet" (see the "Streisand effect"). We further do not allow the seller to modify review contents (see

review perfectly reveals the high quality of the product, guaranteeing that all future consumers of any type will buy the product, and is thus never censored by a seller. Therefore, the seller only faces a nontrivial choice when a bad review arrives. We denote by $r^\theta(h_t)$ the probability with which seller of type θ reveals (or discloses) a bad review that arrives at history h_t.

Equilibrium Definition

All consumers in our model are short-lived, so their behavior is myopic. The only strategic player is the seller. At every history h_t he maximizes his continuation value (21) given the consumers' behavior, and the latter only depends on their current and future beliefs $p := (p^n, p^s)$. Given that all available information about future beliefs is contained in current beliefs and the seller's strategy, and that the seller observes all the information that the consumers see, current beliefs $p(h_t)$ are a sufficient statistic of history h_t. Therefore, we can essentially without loss of generality focus on Markov setting with state p and redefine all objects accordingly.[51] For example, the seller's strategy in such a setting is described by $r^\theta(p_t) = r^\theta(p(h_t)) := r^\theta(h_t)$.

We are then looking for Markov Perfect Equilibria of the game, which consist of a strategy profile $\left(r^H, r^L\right)$ and updating rules for beliefs p such that

1. at any state p strategy r^θ maximizes $V^\theta(p)$ for $\theta \in \{L, H\}$ given the updating rules for p_t and the initial condition $p_0 = p$;

2. beliefs p are updated given strategies $\left(r^H, r^L\right)$ in such a way that

    - p^s is updated using Bayes' rule whenever possible;
    - p^n is updated using Bayes' rule whenever possible under the assumption that $r^H(p) = r^L(p) = 1$ for all p;
    - $p^\gamma = 0$ at histories that a consumer of type γ perceives as being off the equilibrium path.

The latter condition about off-path histories is made purely for convenience and is without loss of generality: if there exists an equilibrium with some off-path beliefs, it can as well be sustained by the most pessimistic off-path belief.

Section II for references to papers that do).

[51] Formally, some equilibria are lost as a result of the restriction to Markov strategies. In particular, many states p admit multiple possible continuation equilibria. In this case we lose [Perfect Bayesian] equilibria which prescribe different continuations at different histories h_t which generate the same state p_t. However, this is not a meaningful loss in the sense that any payoff profile attainable in some PBE of our model can also be generated by some MPE augmented by a public randomization device.

Equilibrium Analysis

This section contains the characterization of equilibria of the game, which culminates in the two main results. Formal proofs of all statements can be found in the Appendix. We start, however, with discussing some preliminaries.

Preliminaries: Multiplicity

Like most communication games, our model features a multiplicity of self-fulfilling equilibria.[52] The loop for any given state p proceeds as follows: if no bad reviews are ever revealed at p, then sophisticated consumers can have arbitrary beliefs in case a bad review *is* revealed at p. In particular, consumers can ascribe $f^s(p) = 0$ after such an off-path event, which makes revealing a bad review at p a weakly dominated action for any seller because the naives' demand cannot increase after a bad review, thus closing the loop. Using this reasoning, we can "ban" disclosure of bad reviews on any subset of the state space.[53] We do not refine such situations away, but our main interest lies in the regions where bad reviews are disclosed.

One particular equilibrium deserves special attention:

Definition 6. *An equilibrium is* fully censored *if* $\left(r^H(p), r^L(p)\right) = (0,0)$ *for all* $p \in [0,1]^2$.

In the fully censored equilibrium, all bad reviews are always deleted. This equilibrium is special in the sense that it always exists, as the verbal reasoning above implies. One of the main contributions of our paper is showing that equilibria exist that are not fully censored (and not payoff-equivalent to fully censored equilibrium), i.e., bad reviews are revealed in a payoff-relevant way in such equilibria. In other words, censorship is a trivial phenomenon in equilibrium; it is the *lack* of censorship that is not trivial.

To make the classification of equilibria easier, consider the following piece of notation. Given some equilibrium, let $\mathcal{R} \subseteq [0,1]^2$ denote the set of states in which bad reviews are revealed on equilibrium path with positive probability: $\mathcal{R} := \{p \mid \left(r^H(p), r^L(p)\right) \neq (0,0)\}$. Then after observing a bad review at some $p \in \mathcal{R}$ the sophisticated consumer updates her belief p^s using Bayes' rule, while a bad review at $p \notin \mathcal{R}$ leads to $f^s(p) = 0$ according to our off-path updating rule. The fully censored equilibrium is characterized by $\mathcal{R} = \emptyset$.

[52] "Self-fulfilling equilibrium" is used as a heuristic notion and not in the sense of any formal definition.

[53] More generally, bad reviews can be banned at arbitrary sets of histories h_t of the game, leading to some non-Markov equilibria.

Preliminaries: Beliefs

This section explores how beliefs $p = (p^n, p^s)$ are updated throughout the game. At any given history, one of three mutually exclusive events can happen: a good review is posted, i.e., written by a consumer and not deleted by the seller, a bad review is posted, or no new reviews are posted. After any single good review the product is revealed to be good, and beliefs of both types of consumers jump to $p^n = p^s = 1$. Conditional on the other two events, the two types of consumers update their beliefs differently.

Recall that $f^\gamma(p)$ denotes the posterior belief of a consumer of type γ who has observed a bad review posted in state p. For a sophisticated consumer, Bayes' rule prescribes that the belief is updated as

$$\frac{f^s(p)}{1 - f^s(p)} = \frac{p^s}{1 - p^s} \cdot \frac{(1-q) \cdot r^H(p)}{r^L(p)}. \tag{12}$$

A naive consumer uses the same Bayes' rule to update her beliefs but under the assumption that $r^\theta(p) = 1$ for both θ. Therefore, her belief is updated as

$$\frac{f^n(p)}{1 - f^n(p)} = \frac{p^n}{1 - p^n} \cdot (1-q) \,. \tag{13}$$

Note that the right-hand side does not depend on p^s or equilibrium strategies $r^\theta(p)$. The $(1-q)$ term in (12) and (13) is the "inherent meaning" of a bad review – the fact that absent any other information, the belief should decrease. The $r^H(p)/r^L(p)$ ratio in (12) represents the information about quality θ contained in the seller's strategies.

Finally, if no reviews are published during $[t, t+dt)$ then sophisticated consumers update their beliefs as

$$\frac{p^s_{t+dt}}{1 - p^s_{t+dt}} = \frac{p^s_t}{1 - p^s_t} \cdot \frac{(1 - \lambda\pi(p)dt) + \lambda\pi(p)dt \cdot (1-q) \cdot \left(1 - r^H(p_t)\right)}{(1 - \lambda\pi(p)dt) + \lambda\pi(p)dt \cdot (1 - r^L(p_t))}.$$

By the usual argument, which involves taking logarithms of both sides and using the approximation $\ln(1+x) \approx x$ for small x, we obtain

$$\dot{p}^s = \lambda p^s \left(1 - p^s\right) \pi(p) \cdot \left[(1-q) \cdot \left(1 - r^H(p)\right) - \left(1 - r^L(p)\right)\right].$$

Hereinafter we use a shorthand notation for the drift term $D(p) := (1-q) \cdot \left(1 - r^H(p)\right) -$

$\big(1 - r^L(p)\big)$, so the expression above can be written as

$$\dot{p}^s = \lambda p^s \left(1 - p^s\right) \cdot \pi(p) D(p). \tag{14}$$

For naive consumers, the similar procedure under the assumption $r^H(p) = r^L(p) = 1$ yields $\dot{p}^n = 0$. Since the intensity λ of reviews' arrival in the absence of censorship is the same for high- and low-quality products, the lack of reviews is uninformative for naive consumers, and their belief stays frozen until a new review is published.

Preliminaries: Bands and Patience

In the analysis it will prove useful to have measures of demand for the two groups of consumers. Demand here is understood not in the sense of "how much a given consumer buys" but rather "how long it takes until type-γ consumers stop buying the product." We introduce these measures of "how long" in different ways for naive and sophisticated consumers.

Belief p^n of naive consumers in the absence of good reviews is fully determined by the prior and the number of posted bad reviews. In particular, it is independent of time, so naive consumers do not change their purchasing behavior as long as no new reviews are posted. We can, however, keep track of how many more bad reviews they are ready to observe in the absence of good reviews before they stop buying the product. We do this by partitioning the state space (p^n, p^s) into "bands" $\{\mathcal{B}_k\}_{k \geqslant 0}$, where k corresponds to the number of bad reviews needed to drop the naive consumers' belief p^n below the threshold $\bar{p}$. In particular, let $\mathcal{B}_0 := \{p \mid p^n \in [0, \bar{p})\}$ and for $k \in \mathbb{N}$ let $\mathcal{B}_k := \{p \mid f(p) \in \mathcal{B}_{k-1}\} \backslash \mathcal{B}_{k-1}$. Additionally, let $\mathcal{B}_{k+} := \cup_{k' \geqslant k} \mathcal{B}_{k'}$. We further split each band into two parts according to whether p^s is above or below $\bar{p}$, with $\mathcal{B}_k^{\uparrow} := \mathcal{B}_k \cap \{p^s \geqslant \bar{p}\}$ and $\mathcal{B}_k^{\downarrow} := \mathcal{B}_k \cap \{p^s < \bar{p}\}$.

Figure 4 illustrates bands together with an example path of beliefs from time zero until a point where all consumers stop buying the product in the absence of a good review. Solid arrows show how beliefs drift over time in the absence of reviews, dotted arrows show how beliefs jump when a bad review is disclosed, and $\bar{\bar{p}}$ is such that $f^n(\bar{\bar{p}}, p^s) = \bar{p}$ for arbitrary p^s.

On the other hand, to measure sophisticated consumers' demand we use their "patience" $\tau(p)$ – the time it takes sophisticated consumers to quit the market in the absence of any reviews. Formally, for any k and all $p \in \mathcal{B}_k$, let

$$\tau(p) := \inf\{t \mid p_t \in \mathcal{B}_k^{\downarrow},\, p_0 = p\},$$

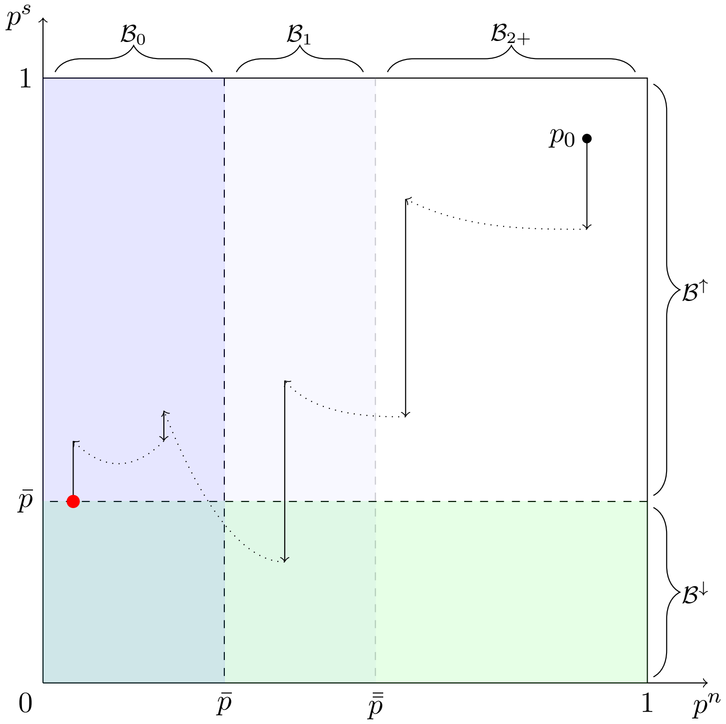

Figure 4: state space, "bands" and an example path of beliefs

where the evolution of p_t is given by (14).[54] Let $\tau(p) = +\infty$ if p_t never reaches $\mathcal{B}_k^{\downarrow}$ from p.

Since "deleting all future bad reviews" is an available strategy, $\tau(p)$ gives a lower bound on time for which a seller can keep sophisticated consumers in the market starting from p. As we will see in Section II, the low-type seller can actually do no better than this, meaning that $\tau(p)$ is the exact measure of residual demand from sophisticated consumers faced by the low-type seller.

Two important properties of $\tau(p)$ are worth noting. Firstly, it follows directly from (14) (see Lemma 11 in the Appendix) that for any $p \in \mathcal{B}_k^{\uparrow}$ for some k, $\tau(p)$ can be expressed as

$$\tau(p) = -\int_{\bar{p}}^{p^s} \frac{1}{\lambda z\,(1-z)\cdot \pi(p^n, z) D(p^n, z)} dz.$$

Secondly, we claim that $\tau(p)$ is actually finite for interior p^s:

Lemma 1. *In any equilibrium, $\tau(p) < +\infty$ and, thus, $D(p) < 0$ for all p with $p^s < 1$.*

The formal proof is contained in the Appendix, although the intuition behind it is simple. Assume by way of contradiction that there exists p such that $\tau(p) = +\infty$. Then once in state p,

[54] To clarify, $\tau(p)$ is not the "calendar time" of the game when it reaches $\mathcal{B}_k^{\downarrow}$, but rather the time it takes beliefs to drift from p to $\mathcal{B}_k^{\downarrow}$. If the origin state p was reached at some time t then $\mathcal{B}_k^{\downarrow}$ will be reached at time $t + \tau(p)$ in the absence of posted reviews. This interpretation of time is used throughout the paper, and calendar time is ignored throughout.

the seller is able to retain sophisticated consumers forever by deleting all future bad reviews. On the other hand, $\tau(p) = +\infty$ implies that deleting bad reviews will bring the seller either to a state with zero drift, or to an interval of states that requires infinite time to drift through. In either case, this means that for any small $\varepsilon > 0$ there exists some state $\hat{p} = (\hat{p}^n, \hat{p}^s)$ with: (1) $D(\hat{p}) \geqslant -\varepsilon$, and (2) $\tau(\hat{p}) \geqslant 1/\varepsilon$. The former property implies that some type of the seller should be willing to reveal a bad review at $\hat{p}$ (since otherwise $D(\hat{p}) = -q$ by (14)), and that $f^s(\hat{p}) < \hat{p}^s$ by the martingale property of beliefs.[55] The latter fact – that $\tau(\hat{p})$ is effectively infinite – implies that revealing a bad review should also retain sophisticated consumers for an arbitrarily long time (i.e., $\tau(f(\hat{p})) \approx +\infty$), as otherwise neither type of the seller would have any incentives to reveal (since a bad review cannot increase sales to naive consumers). However, then we arrive at a state $f(\hat{p})$ with $f^s(\hat{p}) < \hat{p}^s$ and arbitrarily large $\tau(f(\hat{p}))$, so the same argument can be applied again. By iterating the procedure we are bound to eventually arrive at a state with $p^s < \bar{p}$ where the sophisticated consumers no longer buy the product. This leads to a contradiction, since in that state $\tau(p) = 0$.

Main Results

The remainder of Section II is devoted to characterizing the equilibria of the game. Sections II to II provide a detailed characterization, while the current section summarizes the main results and provides a condensed version of the intuition behind them.

The first main result, Theorem 2, states that if sophisticated consumers are buying the product, then any bad review they observe when ready to buy the product will weakly improve their belief in product quality. In particular, this implies that if a sophisticated consumer is willing to buy the product after observing current time, then reading bad reviews cannot change her mind. We dub this result "reversal", since sophisticated consumers' reaction to bad reviews is reversed from its natural direction – instead of decreasing p^s, any bad review manages to increase it.

Theorem 2 (Reversal)**.** *In any equilibrium of the game with $\mathcal{R} \neq \emptyset$: if $p \in \mathcal{R}$ and $p^s \geqslant \bar{p}$ then $f^s(p) \geqslant p^s$.*

Condition $p^s \geqslant \bar{p}$ ensures that the sophisticated consumers' opinion actually matters. This is a sufficient condition for reversal in our model but not a necessary one (as we see below, $f^s(p) > p^s$ for $p \in \mathcal{B}_1^{\downarrow} \cap \mathcal{R}$, even though by definition $p^s < \bar{p}$ for all $p \in \mathcal{B}_1^{\downarrow}$). Further, the theorem only has

[55] Given that p^s strictly increases after a good review, it has to go down either after a bad review, or in the absence of reviews. We show that $D(p) = -q$ is required for $f^s(p) = p^s$ (see Lemma 10 in the Appendix), while if $D(p) = -\varepsilon > -q$, then it should be that $f^s(p) > p^s$.

any power when looking at equilibria in which bad reviews are revealed – although the statement itself formally holds for the fully censored equilibrium as well.

While Theorem 2 may seem trivial at first – "if a seller reveals bad reviews then it must be profitable for him to do so" – the devil, as per the tradition, is in the details. In particular, it is not obvious that "more profitable" corresponds to $f^s(p) > p^s$, since the latter property does not guarantee $\tau(f(p)) > \tau(p)$. Before discussing the intuition behind Theorem 2, we state one of its corollaries, which claims that in any equilibrium, the high-type seller is less likely to conceal any bad review than the low-type seller.

Corollary 1. *In any equilibrium of the game with $\mathcal{R} \neq \emptyset$: if $p \in \mathcal{R}$ and $p^s \geqslant \bar{p}$ then $r^H(p) > r^L(p)$.*

This corollary is an immediate consequence of equation (12) and Theorem 2. While formally trivial, this corollary is valuable in that it describes the mechanics of Theorem 2: reversal is achieved via the low-type seller deleting more bad reviews than the high type. The fact that a bad review was *not* deleted is then a strong signal of high quality, which in the end outweighs the inherently negative information contained in the review.

The reasoning behind Theorem 2 proceeds in two steps. First we show that in any equilibrium the low-type seller must be indifferent between revealing and deleting a bad review at any $p \in \mathcal{R}$ (see Lemma 2 below). Then we show that any strategy profile that satisfies this indifference also necessarily satisfies the first statement of the Theorem.

Arguably the more interesting part of the proof is the second step: from indifference to $f^s(p) \geqslant p^s$. This result comes through two main channels: the compensation effect and the expectancy effect. The *compensation effect* states that if $p \in \mathcal{B}_1$, i.e., $p^n \geqslant \bar{p} > f^n(p)$ – naive consumers are close to quitting the market and one more bad review drives them out, – then the seller's decision to disclose a bad review should be rewarded by higher demand from sophisticated consumers. This comes from equilibrium reasoning: if a bad review is disclosed then it is beneficial to do so for some type of the seller, meaning that if the seller loses naive consumers, demand from sophisticated consumers should increase. This higher demand requirement then translates to an increase in reputation requirement.

The *expectancy effect* is more subtle and can be seen as ripples on the water, propagating the original effect away from $\mathcal{B}_1$ into the $\mathcal{B}_{2+}^{\uparrow}$ region. By the martingale property of beliefs, values of $f^s(p) - p^s$ and $D(p)$ are negatively associated for any given p.[56] Therefore, the situation in $\mathcal{B}_1$ creates very high *expectancy* for sophisticated consumers; either outcome affects their belief significantly.

[56] We use "negatively associated" as an informal term; its exact meaning is given by Lemma 10 in the Appendix.

Any bad review that is revealed improves it by a lot, but in the absence of reviews this belief deteriorates rapidly. In particular, high expectancy makes sophisticated consumers impatient: for a given p, more expectancy in the near future leads to lower patience $\tau(p)$, which is disliked by the seller. Therefore, in order to incentivize the seller to reveal bad reviews in $\mathcal{B}_2^{\uparrow}$ – and expose himself to this state of high expectancy, – the seller should be rewarded with a reputation premium for doing so. This premium, in turn, increases expectancy above baseline in $\mathcal{B}_2^{\uparrow}$, and the whole reasoning unravels to bands $\mathcal{B}_k^{\uparrow}$ with $k > 2$. Noteworthy is the fact that if expectancy in $\mathcal{B}_1^{\uparrow}$ is high enough to start this chain reaction, then *strictly* positive reputation premia are required in $\mathcal{B}_{2+}^{\uparrow}$ to incentivize the seller to reveal a bad review – even though this does not lead to immediate loss of naive consumers' demand.

Our second main result, Theorem 3, demonstrates existence of equilibria in which bad reviews are revealed in a payoff-relevant way. It claims that there exist equilibria that

1. are non-trivially different from the fully censored equilibrium in terms of payoff, and
2. admit strict reversal: $f^s(p) > p^s$ for all $p \in \mathcal{R}$.

We construct one family of equilibria that exhibit both features, but one can easily construct an equilibrium that has one of the above features and not the other.

Theorem 3 (Revelation)**.** *In the set of all equilibria for any given parameter values*

1. *if $\mu \in [0, 1/2]$, then all equilibria are payoff-equivalent to the fully censored equilibrium;*
2. *if $\mu \in (1/2, 1)$, then there exist equilibria with $\mathcal{R} \neq \emptyset$, which have $f^s(p) > p^s$ for all $p \in \mathcal{R}$, and which are not payoff-equivalent to the fully censored equilibrium.*

The first statement of Theorem 3 follows from Proposition 11 below, which implies that if $\mu < 1/2$, then $\mathcal{R} \cap \mathcal{B}_1 = \emptyset$. Therefore, naive consumers' demand cannot be affected by any sequence of bad reviews in equilibrium – bad reviews can only be revealed in $\mathcal{B}_0$ and $\mathcal{B}_{2+}$, where p^n and $f^n(p)$ are always on the same side of $\bar{p}$. In other words, bad reviews can never work as a costly signal because they are never actually costly in terms of driving naive consumers out of the market. Sophisticated consumers then ignore bad reviews altogether. In the end, while some bad reviews may be revealed in equilibrium, they do not have any payoff-relevant effects.

The second statement of Theorem 3 is ex ante not trivial. Basic models of disclosure (such as Grossman [1981] and Jung and Kwon [1988]) predict that revealing bad news to a sophisticated audience is always suboptimal. It is straightforward that revealing bad news to a purely naive audience is also suboptimal. However, Theorem 2 shows that the presence of naive consumers in the market affects sophisticated consumers' reaction to bad news, rendering it positive.[57] The main message of Theorem 3 is that this reversal can justify the revelation of bad news as long as naive consumers do not dominate the market. Furthermore, the value of Theorem 3 is in saying that bad news can be revealed in a payoff-relevant way. This is in contrast to, e.g., Grossman [1981], where the lowest type may disclose his information in equilibrium, but this would be equivalent to an equilibrium in which bad news are never revealed.

The main idea behind Theorem 3 is as follows. Reversal – which is satisfied by all equilibria as per Theorem 2, and hence is a necessary condition for equilibrium – requires that the high-type seller reveals more bad reviews than the low type (Corollary 1). Therefore, the high type must be weakly more willing to reveal bad reviews at all $p \in \mathcal{R}$. We show that this preference indeed exists when $\mu > 1/2$ because the high type faces higher rate of arrival of good reviews and is thus less afraid to lose naive consumers to bad reviews than the low type.

The remainder of Section II characterizes the game's equilibria in greater detail. A reader who is not interested in these details is invited to skip to Section II. Section II argues that the low-type seller has to always be indifferent between revealing and deleting bad reviews, and demonstrates the implications of this indifference for equilibrium strategy profiles and belief dynamics. Theorem 2 relies on Section II only. Section II explores the incentives of the high-type seller conditional on low type's indifference. Section II describes an example of the equilibrium that satisfies the conditions of Theorem 3.

Characterization: Low Type's Preferences

The first big step in understanding the equilibria of the game relates to incentives of the low-type seller. In particular, we show that for any bad review that can be revealed in equilibrium, the low-type seller must be indifferent between revealing this bad review and deleting it.

Lemma 2. *In any equilibrium, all $r^L(p) \in [0,1]$ are optimal at all $p \in \mathcal{R}$. Consequently, deleting all future bad reviews is an optimal continuation strategy for the low-type seller at all p. Furthermore:*

1. $\tau(p) = \tau(f(p))$ *for all* $p \in \mathcal{B}_0^{\uparrow} \cap \mathcal{R}$,
2. $e^{-r\tau(p)} = \frac{1-\mu}{\mu} + e^{-r\tau(f(p))}$ *for all* $p \in \mathcal{B}_1^{\uparrow} \cap \mathcal{R}$,

[57] Recall that we only allow $\mu \in [0, 1)$ in the model. It is argued in Section II that in case $\mu = 1$, all equilibria are payoff-equivalent to the fully censored equilibrium. Therefore, naive consumers are necessary for bad reviews to be revealed in a payoff-relevant way.

3. $\tau(p) = \tau(f(p))$ *for all* $p \in \mathcal{B}_{2+}^{\uparrow} \cap \mathcal{R}$.

The intuition behind this result is best understood from reasoning by contradiction. Fix arbitrary $p \in \mathcal{R}$. If out of the two actions (deleting a bad review and not) at p the low type only ever does one but not the other, then "the other" becomes a strong positive signal for sophisticated consumers – so strong that the low-type seller should always find it strictly optimal to pick the "other" action. The details of the argument differ for the two actions, but the essence boils down to the reasoning above.

The behavioral strategy $r^L(p) = 0$ (deleting a bad review at p for sure) is weakly optimal for low-type seller at any $p \in \mathcal{R}$ by Lemma 2 and strictly optimal at any $p \notin \mathcal{R}$ due to the assumption that $f^s(p) = 0$ for $p \notin \mathcal{R}$. Therefore, deleting all bad reviews is trivially a weakly optimal continuation strategy. In this case $V^L(p)$ equals the discounted profit from deleting all future bad reviews:

$$V^L(p) = \frac{1}{r}\left[(1-\mu)\cdot \mathbf{1}(p^n \geqslant \bar{p}) + \mu \cdot \left(1 - e^{-r\tau(p)}\right)\right]. \tag{15}$$

In particular, notice that $V^L(p)$ only depends on $\tau(p)$ and the indicator $\mathbf{1}(p^n \geqslant \bar{p})$.

Finally, given the optimality of deleting all future bad reviews, the equalities in Lemma 2 follow directly by ensuring that $V^L(p) = V^L(f(p))$ for all $p \in \mathcal{R}$. The patience of sophisticated consumers should increase in such a way as to exactly compensate for the loss of naive consumers from revealing a given bad review. In particular, it should be unchanged if the purchasing behavior of naive consumers is unaffected by the revelation.

We now move on to exploring the implications of Lemma 2 for belief dynamics. We essentially unravel the game by backward induction on the state space, analyzing different regions separately.

Band $\mathcal{B}_0$

In $\mathcal{B}_0 = \{p | p^n \in [0, \bar{p})\}$, naive consumers are too pessimistic about the product quality to make a purchase. If in addition $p^s < \bar{p}$ (i.e., $p \in \mathcal{B}_0^{\downarrow}$), then the same applies to sophisticated consumers, and the market collapses – no purchases are made and no reviews are written. Region $\mathcal{B}_0^{\downarrow}$ is thus an absorbing state and serves as a starting point for the "unraveling" of the state space.

If $p^s \geqslant \bar{p}$ (that is, $p \in \mathcal{B}_0^{\uparrow}$), then only sophisticated consumers buy the product. Since p^n is frozen absent any reviews, only two escapes are possible from $\mathcal{B}_0^{\uparrow}$: either a good review is posted and consumers' beliefs jump to $p^n = p^s = 1$, in which case the seller's strategy becomes irrelevant and all consumers stay in the market forever, or the sophisticated consumers become too pessimistic and the game arrives at the region $\mathcal{B}_0^{\downarrow}$ described above (from Lemma 1, we know this happens in finite time). This structure allows us to characterize continuation equilibria of the game starting from any state $p \in \mathcal{B}_0^{\uparrow}$.

Proposition 10 states that disclosure of a bad review should not affect the belief of sophisticated consumers in $\mathcal{B}_0^{\uparrow}$. Whenever naive consumers have quit the market, the seller can no longer signal his credibility to the sophisticated consumers by sacrificing naive consumers' demand.

Proposition 10. *Strategy profile $\left(r^H(p), r^L(p)\right)$ constitutes an equilibrium if and only if $f^s(p) = p^s$ for all $p \in \mathcal{B}_0^{\uparrow} \cap \mathcal{R}$.*

From Lemma 2 we already know that since revealing a bad review at $p \in \mathcal{B}_0^{\uparrow} \cap \mathcal{R}$ does not affect the purchasing behavior of naive consumers, it should also have no effect on sophisticated consumers' patience: $\tau(p) = \tau(f(p))$. It is relatively straightforward that an equilibrium with $f^s(p) = p^s$ for all $p \in \mathcal{B}_0^{\uparrow} \cap \mathcal{R}$ would satisfy this requirement. The value of Proposition 10 hence lies in showing that the converse is also true: sophisticated consumers' belief *has* to stay unaffected in order to warrant $\tau(p) = \tau(f(p))$, since in any other case the equilibrium cannot be sustained. This also implies that all equilibria are payoff-equivalent in $\mathcal{B}_0$.

Corollary 2. *All continuation equilibria starting from any given $p \in \mathcal{B}_0$ are payoff-equivalent to the fully censored continuation equilibrium.*

This Corollary follows from the fact that at any given p, drift speed $D(p)$ is the same whether $p \notin \mathcal{R}$ or $p \in \mathcal{R}$ and $f^s(p) = p^s$ – if bad reviews are irrelevant for sophisticated consumers then it does not matter whether they are revealed in $\mathcal{B}_0$ or not.

It is worth noting that Proposition 10 and Corollary 2 apply to the whole state space (i.e., all histories) in case $\mu = 1$ when all consumers are sophisticated. While this case is purposefully not included in the model setup (all following results only apply to $\mu \in [0, 1)$), this is mostly for the sake of narrative clarity. The fact that Proposition 10 applies globally when $\mu = 1$ means that in the absence of naive consumers, bad reviews are completely irrelevant under censorship. Even though some bad reviews may be revealed in that case (since $f^s(p) = p^s$ requires $r^L(p) < 1$ for any $p \in \mathcal{R}$ as per (12)), those that *are* revealed carry no useful information whatsoever and have no effect on [sophisticated] consumers' behavior. Furthermore, revealing bad reviews in that setting has no value to either the seller or the consumers, so all equilibria are payoff-equivalent to the fully censored equilibrium.[58] Therefore, all following results rely on nonzero market presence of naive

[58]The key to observing this equivalence is noting that $D(p) = -q$ both if $p \notin \mathcal{R}$ and if $p \in \mathcal{R}$ and $f^s(p) = p^s$.

consumers.

Band $\mathcal{B}_1$

Continuing to band $\mathcal{B}_1 = \{p \mid p^n \in [\bar{p}, \bar{\bar{p}})\}$, one should notice that starting from $\mathcal{B}_1$ the beliefs may only jump to $p^n = p^s = 1$ after a good review, band $\mathcal{B}_0$ after a bad review, or stay in $\mathcal{B}_1$ forever absent any reviews. Therefore, having full knowledge of continuation equilibria in $\mathcal{B}_0$, we can describe the continuation equilibria that start in $\mathcal{B}_1$. The parts of most interest are given by the two following propositions. Proposition 11 states that bad reviews are only revealed in $\mathcal{B}_1$ if there are sufficiently many sophisticated consumers in the market ($\mu \geqslant 1/2$) and they are sufficiently close to quitting the market (p^s is low enough). Proposition 12 then concludes that whenever a bad review *is* disclosed in $\mathcal{B}_1$, it strictly improves sophisticated consumers' belief p^s.

Proposition 11. *For any $p \in \mathcal{B}_1$: $p \in \mathcal{R}$ only if $\mu \geqslant 1/2$ and $p^s < p^*$, where*

$$p^* = \frac{\bar{p}\mu^{\frac{\lambda}{r}}}{\bar{p}\mu^{\frac{\lambda}{r}} + (1-\bar{p})(1-\mu)^{\frac{\lambda}{r}}}.$$

Proposition 12. *In any equilibrium of the game $f^s(p) > p^s$ for all $p \in \mathcal{B}_1 \cap \mathcal{R}$.*

By Lemma 2 the low-type seller should be indifferent between revealing and deleting a bad review at any $p \in \mathcal{B}_1 \cap \mathcal{R}$. He can retain naive consumers in the market forever starting from any p with $p^n \geqslant \bar{p}$ and can never bring them back to the market starting from any p with $p^n < \bar{p}$. Revealing a bad review at some $p \in \mathcal{B}_1$ and losing naive consumers forever is then always worse than deleting it and retaining naive consumers forever – even if revealing the review brings sophisticated consumers to the market and retains them forever. Therefore, any reason to allow bad reviews in $\mathcal{B}_1$ only arises if $\mu \geqslant 1/2$, i.e., if sophisticated consumers are more prevalent in the market than naive consumers. This gives the first condition in Proposition 11.

The second condition in Proposition 11 comes from the fact that at some states $p \in \mathcal{B}_1^{\uparrow}$ patience $\tau(p)$ is so large that even a jump to the most optimistic belief $f^s(p) = 1$ – which grants the seller sales to sophisticated consumers from now until eternity – is not sufficient to compensate the seller for the loss of naive consumers. This leads to an upper bound on p^s at which bad reviews may be disclosed.

The fact that any revealed bad review trades off naive consumers' demand for that of sophisticated consumers is the basic idea behind Proposition 12: $\tau(p)$ should increase following disclosure, which implies that p^s should increase as well. This implication is not as trivial as may seem at first because the speeds at which the belief of sophisticated consumers drifts toward $\bar{p}$ in $\mathcal{B}_0$ and $\mathcal{B}_1$ are not the same in general.

Band $\mathcal{B}_{2+}$

Analogous to before, from $\mathcal{B}_2$ the beliefs may only escape to $p^n = p^s = 1$ after a good review, into $\mathcal{B}_1$ after a bad review, or else stay in $\mathcal{B}_2$ if no reviews are posted. Therefore, we can apply our knowledge of continuation equilibria in $\mathcal{B}_1$ to explore the continuation equilibria originating in $\mathcal{B}_2$ and then unravel to include $\mathcal{B}_k$ with $k > 2$. This will conclude our analysis of the implications of the low type's incentives, since the set $\cup_{k\geqslant 0}\mathcal{B}_k$ covers the whole state space, so any equilibrium of the game is a continuation equilibrium starting in one of these bands.

Proposition 13. *In any equilibrium of the game $f(p) \geqslant p^s$ for all $p \in \mathcal{B}_{2+}^{\uparrow} \cap \mathcal{R}$.*

While Proposition 13 may look very similar to Proposition 12, the reasoning behind it is more subtle. An extremely informal explanation follows.

By the martingale property of beliefs, values of $f^s(p) - p^s$ and $D(p)$ are negatively associated: the stronger is the positive reaction to a bad review, the stronger should be the negative reaction to the absence of reviews.[59] We use the informal term "expectancy" to denote the common factor underlying both values: high expectancy is associated with high drift speed $|D(p)|$ (i.e., very negative $D(p)$) and strong reversal $f^s(p) - p^s$, and vice versa.

Recall from Lemma 2 that the low-type seller must be indifferent between disclosing a bad review and deleting it in all $p \in \mathcal{B}_{2+} \cap \mathcal{R}$, so it must be that $\tau(p) = \tau(f(p))$. We know from Proposition 12 that $f^s(p) - p^s > 0$ for all $p \in \mathcal{B}_1 \cap \mathcal{R}$, so expectancy is high in $\mathcal{B}_1$ and hence $D(p)$ is strongly negative there. This means that sophisticated consumers are relatively impatient – $\tau(f(p))$ is "small" for $f(p) \in \mathcal{B}_1^{\uparrow}$. Therefore, the expectancy in $\mathcal{B}_2^{\uparrow}$ should also be high. If it is not then $\tau(p)$ is "large" for $p \in \mathcal{B}_2^{\uparrow}$, while $f^s(p) - p^s$ is "small", which together with sophisticated consumers' impatience in $\mathcal{B}_1$ implies that $\tau(f(p))$ is "very small" – a contradiction to $\tau(p) = \tau(f(p))$. Iterating the same argument over bands, we then get that $f^s(p) \geqslant p^s$ for all $p \in \mathcal{B}_{2+}^{\uparrow} \cap \mathcal{R}$.

The argument above is, admittedly, not very believable without defining what "small" and "large" mean. It also largely ignores the possibility that the revelation set $\mathcal{R}$ may look very differently in the two bands (and the expectancy is different in and out of $\mathcal{R}$ – there is no reason to grow very pessimistic in the absence of reviews if bad reviews are effectively banned at $p \notin \mathcal{R}$),

[59] Although drift speed $|D(p)|$ and degree of reversal $f^s(p) - p^s$ are not connected one-to-one, what matters for the argument is that $D(p) \geqslant (>) - q$ if and only if $f^s(p) - p^s \leqslant (<) 0$ (see Lemma 10).

as well as the fact that even different states in a given $\mathcal{B}_k \cap \mathcal{R}$ can have different expectancy. The formal proof in the appendix addresses all these issues properly.

Unlike in $\mathcal{B}_1$, we cannot make any hard statements about the area below the cutoff, $\mathcal{B}_{2+}^{\downarrow}$. The reasoning for $p \in \mathcal{B}_{2+}^{\uparrow} \cap \mathcal{R}$ relies on the fact that the cutoff $\bar{p}$ is always reached in finite time from any $p^s < 1$. Under the cutoff there is no such terminal point. All states under the cutoff are inherently similar – they all can warrant the status quo forever, where the sophisticated consumers are out of the market until a good review arrives and the naive consumers buy the product forever. Therefore, as long as the seller is guaranteed to arrive at a state with $\tau(p) = 0$ (i.e., $p^s < \bar{p}$), he does not care about the exact $f^s(p)$. This means that there is no problem in having arbitrary jumps of p^s. On the other hand, this also means that there are no impediments to constructing specific equilibria in which $f^s(p) > p^s$ for all $p \in \mathcal{B}_{2+}^{\downarrow} \cap \mathcal{R}$.

Characterization: High Type's Preferences

This section investigates the high-type seller's preferences conditional on the low type's indifference. In doing so we retrace the same path over bands that we followed in the second step of the proof of Theorem 2. For $\mathcal{B}_0^{\uparrow}$, Corollary 2 states that all continuation equilibria at any $p \in \mathcal{B}_0^{\uparrow}$ are payoff-equivalent: when all bad reviews are completely uninformative, it does not make much difference whether any of them are revealed on equilibrium path or not.

The key insight into the high type's incentives lies in $\mathcal{B}_1^{\downarrow}$. In any state $p \in \mathcal{B}_1^{\downarrow} \cap \mathcal{R}$, the seller with a bad review in hand faces the following choice. Deleting this and future bad reviews retains naive consumers in the market forever but cannot attract sophisticated consumers. Revealing the review, on the other hand, brings sophisticated consumers to the market for some time $\tau(f(p))$, but drives the naive consumers away. Figure 5 demonstrates this trade-off graphically, showing expected sales per consumer $\pi(p_t)$ as a function of time for the two strategies outlined above. Time zero on the graph corresponds to states p or $f(p)$ respectively. This graph is valid for arbitrary $p \in \mathcal{B}_1 \cap \mathcal{R}$; in the special case of $p \in \mathcal{B}_1^{\downarrow}$ we have $\tau(p) = 0$.

The low-type seller must be indifferent between deleting and revealing a bad review at any $p \in \mathcal{B}_1 \cap \mathcal{R}$, meaning that expected sales in the absence of future reviews should be equal in the two scenarios.[60] Visually, it means that the areas under the two intertemporal demand curves in Figure 5 should be equal (after discounting future sales appropriately). The graph makes it obvious why $\mu \geqslant 1/2$ is necessary to render the low type indifferent in $\mathcal{B}_1$ (and thus generate a strategy profile

[60]It is enough to consider the case when no reviews arrive after p because the low type can never receive good reviews and always weakly prefers to delete bad reviews so deleting all future bad reviews is an optimal strategy.

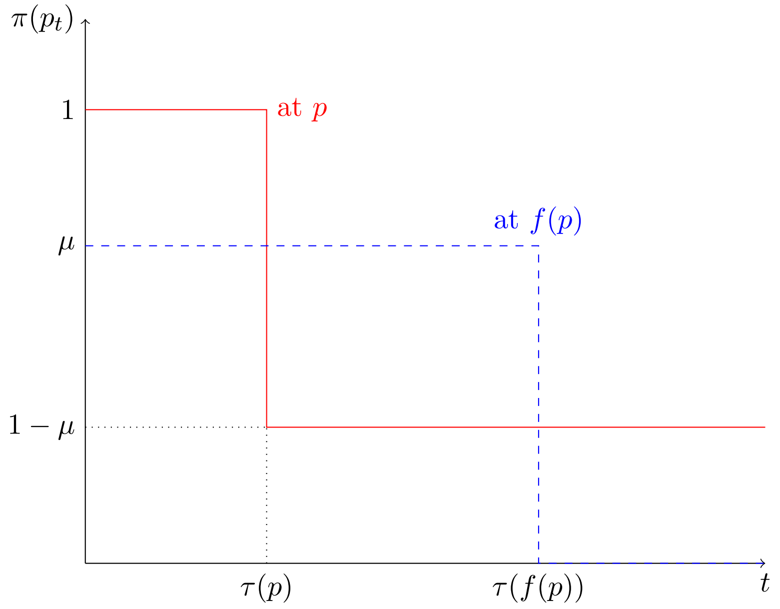

Figure 5: Intertemporal demand starting from some $p \in \mathcal{B}_1 \cap \mathcal{R}$ and $f(p)$.

with nonempty $\mathcal{B}_1 \cap \mathcal{R}$) – otherwise deleting the review and staying at p is strictly better.

We next argue that the high type prefers to reveal at $p \in \mathcal{B}_1 \cap \mathcal{R}$, conditional on the low type's indifference. To see this, note that the only difference between the payoffs of the two types is the option value of receiving a good review for the high-type seller. Because a good review generates the best possible continuation payoff (all consumers buying forever), the high type prefers that this good review arrives sooner rather than later. The rate of arrival of reviews is exactly proportional to sales per consumer. Therefore, conditional on total expected discounted sales being the same in both scenarios (to satisfy the low type's indifference), the high-type seller prefers to frontload sales. In other words, he wants to sell as much product as possible early on in an attempt to generate a good review as early as possible. By looking at Figure 5 it is easy to see that if $\tau(p) = 0$ – which is the case for all $p \in \mathcal{B}_1^{\downarrow}$, – then revealing a bad review and jumping to $f(p)$ generates a more frontloaded demand schedule than deleting a bad review and staying at p. The high type prefers to reveal the bad review because it makes the first good review arrive sooner on average.

At the same time, this can be seen as a costly signaling story. The high-type seller strictly prefers to reveal a bad review and lose naive consumers because this is less costly for him than for the low type. In particular, the high type knows that with positive probability he will receive a good review in the future, which will bring naive consumers back to the market. On the other hand, the fact that revealing a bad review is less costly for the high type than for the low type makes this signal credible for sophisticated consumers, who then react positively to bad reviews.

One can see from Figure 5 that the reasoning for $\mathcal{B}_1^{\downarrow}$ presented above also extends by continuity to $p \in \mathcal{B}_1^{\uparrow} \cap \mathcal{R}$ as long as $\tau(p)$ is low enough. However, it does not need to extend all the way to p^* as given by Proposition 11, meaning that the high type's incentives may provide a tighter bound on p^s for which bad reviews can be revealed in $\mathcal{B}_1$.

Finally, this preference to reveal bad reviews extends from $\mathcal{B}_1$ to $\mathcal{B}_{2+}$ (as long as $\mathcal{B}_1 \cap \mathcal{R}$ is non-trivial). The intuition is as follows. Deleting or revealing a bad review in any $p \in \mathcal{B}_{2+} \cap \mathcal{R}$ has no immediate effect: it affects neither the naive consumers' decision to purchase the product, nor the sophisticated consumers' patience $\tau(p)$ (which again follows from Lemma 2). However, revealing a bad review – and sufficiently many bad reviews after it – will bring the high-type seller to some $p' \in \mathcal{B}_1$. With positive probability he can then receive another bad review in some $p'' \in \mathcal{B}_1 \cap \mathcal{R}$ and reveal it, which, as we have established above, is a strictly preferred option. Thus revealing a bad review at $p \in \mathcal{B}_{2+} \cap \mathcal{R}$ is strictly better than deleting it because it gives the high-type seller a chance to eventually arrive at $\mathcal{B}_0$, which he strictly prefers to staying in $\mathcal{B}_1$ and, by analogy, $\mathcal{B}_{2+}$.

Existence Example

The argument in II implies that the strategy profile akin to the one represented in Figure 6 (the formal construction is in the Appendix) could constitute an equilibrium. In particular, this is exactly the equilibrium that is constructed in the proof of Theorem 3. The orange shaded region is the set $(0,1) \times (0,1) \backslash \mathcal{R}$ where no bad reviews are ever revealed. It includes $\mathcal{B}_0$, most of $\mathcal{B}_1^{\uparrow}$, and the line $\{p \mid p^n \geqslant \bar{\bar{p}}, p^s = \bar{p}\}$. The white region is the revelation set $\mathcal{R} = \{\mathcal{B}_1 \mid p^s < \bar{p}+\varepsilon\} \cup \{\mathcal{B}_{2+} \mid p^s \neq \bar{p}\}$ for some ε. The argument above has shown that the equilibrium conditions (low type's indifference and high type's preference to reveal) can be satisfied for all p within such $\mathcal{R}$.[61] The purple dotted lines show the set of states p that are on equilibrium path given p_0. The blue arrows create a "phase diagram," pointing from p to $f(p)$ for some selected $p \in \mathcal{R}$.

We now want to show that the equilibrium with the revelation set $\mathcal{R}$ described above can satisfy the two properties we are seeking for Theorem 3: it generates payoffs that are different from the fully censored equilibrium and it generates strict reversal in all $p \in \mathcal{R}$. We will start with the latter. We have $\mathcal{B}_0 \cap \mathcal{R} = \emptyset$, and Proposition 12 already gives strict reversal for all $p \in \mathcal{B}_1 \cap \mathcal{R}$. Therefore, it is only left to show strict reversal for $\mathcal{B}_{2+}$. In the case of $p \in \mathcal{B}_{2+}^{\downarrow}$ any action profile can satisfy the equilibrium conditions as long as $f^s(p) < \bar{p}$, so we can easily construct it in such a way that $f^s(p) > p^s$. Finally, the channel through which reversal works in $\mathcal{B}_{2+}^{\uparrow}$ has been described in section II. Importantly, that channel relied on high expectancy (i.e., high drift speeds $|D(p)|$) in $\mathcal{B}_1^{\uparrow}$, which

[61] Incentives for $p \notin \mathcal{R}$ are trivial due to our assumption that off the equilibrium path $p^\gamma = 0$ for either consumer type γ.

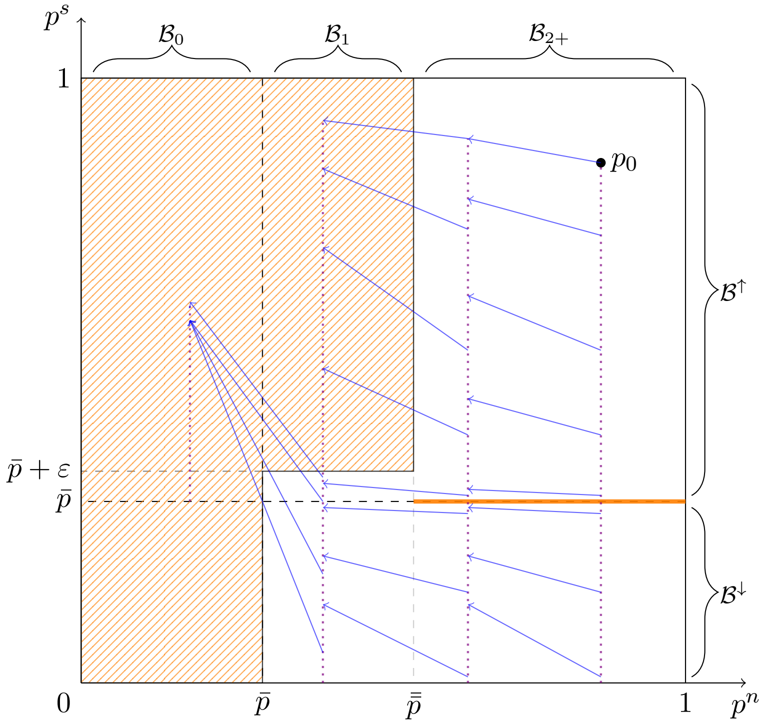

Figure 6: An example of equilibrium with strict reversal.

in turn requires $\mathcal{B}_1^{\uparrow} \cap \mathcal{R} \neq \emptyset$. Therefore, it is important to include the ε-slice of $\mathcal{B}_1^{\uparrow}$ in our revelation set $\mathcal{R}$ to generate strict reversal in $\mathcal{B}_{2+}^{\uparrow}$.

The other property – payoff-nonequivalence – is easy to deduct given the strict reversal. In fully censored equilibrium $D(p) = -q$ for all p. In the equilibrium we have constructed, $f^s(p) > p^s$ for almost all $p \in \mathcal{B}_{2+}^{\uparrow}$, which by Lemma 10 implies that $D(p) < -q$ for all p. Using representation (77) of $\tau(p)$, we immediately obtain that for any given $p \in \mathcal{B}_{2+}$ (with $p^s > \bar{p}$), sophisticated consumers' patience $\tau(p)$ is lower in the equilibrium we have constructed than in the fully censored equilibrium. From Lemma 2 and the consequent optimality of deleting all bad reviews for the low-type seller, one can then conclude that $V^L(p)$ is also lower for such p in the equilibrium of Figure 6 than in the fully censored equilibrium.

Discussion and Extensions

This section presents additional observations resulting from our model, which are not directly related to Theorems 2 and 3. We then consider some extensions of the baseline model.

Seller's Profit

One topic that persists throughout the model is multiplicity of equilibria, which differ in terms of $\mathcal{R}$ – the set of states at which bad reviews are revealed on the equilibrium path. It is then natural to ask which types of players prefer which equilibria. Theorem 3 says that in case $\mu \leqslant 1/2$ all equilibria are payoff-equivalent. Therefore, from this point onward assume that $\mu \in (1/2, 1)$. Proposition 14 below addresses the question of seller's profit. The next subsection discusses issues related to consumers' welfare.

In general, multiple equilibria may exist with the same revelation set $\mathcal{R}$, and payoff comparison across such equilibria is ambiguous. Therefore, we employ the following equilibrium selection.

Definition 7. *An equilibrium* $(r^L(p), r^H(p))$ *is called* semi-separating *if* $r^H(p) = 1$ *for all* $p \in \mathcal{R}$.

This class of equilibria is non-empty, since our constructive proof of the second part of Theorem 3 presents one such equilibrium.[62] As discussed above, Corollary 1 directly implies that at any $p \in \mathcal{R}$ we have $r^H(p) > r^L(p) \geqslant 0$, so the high-type seller should at least weakly prefer to disclose a bad review at p. The refinement above then only rules out the case when the high-type seller is exactly indifferent *and* deletes bad reviews with positive probability.

Proposition 14. *Suppose that* $\mu \in (1/2, 1)$ *and consider two semi-separating equilibria with revelation sets* $\mathcal{R}'$ *and* $\mathcal{R}'' \subset \mathcal{R}'$*, respectively. Then the low-type seller weakly prefers equilibrium with* $\mathcal{R}''$ *to equilibrium with* $\mathcal{R}'$ *at all* p.

Larger $\mathcal{R}$ means more bad reviews are revealed in equilibrium, but it also leads to higher expectancy, making sophisticated consumers less patient. The latter implies the larger is $\mathcal{R}$, the smaller is $\tau(p)$ for all $p \in \mathcal{B}^{\uparrow}$, which in turn makes the low-type seller strictly worse off at those states.

As for the high-type seller, he (weakly) benefits from revealing bad reviews, so he prefers equilibria with larger $\mathcal{B}^{\downarrow}_{1+} \cap \mathcal{R}$ conditional on $f(p)$ being the same in both equilibria for all $p \in \mathcal{B}^{\downarrow}_{1+} \cap \mathcal{R}$. However, in $\mathcal{B}^{\uparrow}_{1+}$ the two effects described above – less-patient consumers given larger $\mathcal{R}$ but more opportunities to reveal a bad review – work in the opposite directions, so the high type's final preferences are ambiguous. Intuitively, in state $p \in \mathcal{B}^{\uparrow}_{1+}$ where both p^n and p^s are small enough, the high-type seller prefers large $\mathcal{R}$ because option value of signalling outweighs the lower patience of sophisticated consumers. On the other hand, by the same reasoning, if either p^n

[62] In fact, if there exists some equilibrium with a given $\mathcal{R}$, then there exists a semi-separating equilibrium with that $\mathcal{R}$.

or p^s is large then the seller evaluating his perspectives in state p would prefer to be in equilibrium with small $\mathcal{R}$. This intuition is confirmed by numerical simulations (python code available upon request).

Asymptotic Learning

The expected utility of a consumer arriving at time t would depend on reviews revealed up to t and is thus tough to measure, even in expectation. However, it is possible to make limit statements. This subsection examines whether the seller's type is learned by the consumers asymptotically as $t \to +\infty$.

Assume that $p_0 > \bar{p}$ to avoid triviality. We say that the seller's type is asymptotically learned by consumers of type γ if the probability that the purchase decision made by consumer of type γ is correct if it approaches 1 as $t \to +\infty$.[63] Asymptotic learning is trivially connected to the welfare of consumers who arrive sufficiently late: if the seller's type is learned, then consumers have full information and thus make efficient purchasing decisions. Not perfectly identifying either type of seller is associated with losses from either buying a low-quality product or not buying a high-quality product.

The bad news for consumers is that learning both sellers' types, by the design of our information structure, is impossible in any equilibrium. To see this, suppose there exists some time t such that absent a good review, both types of consumers stop buying the product by time t with positive probability. With probability $e^{-\lambda qt} > 0$, a high-type seller receives no good review by time t, meaning that with positive probability consumers stop purchasing a high-quality product. On the other hand, if no such time t exists, then consumers never stop buying a low-quality product. Therefore, in any equilibrium at most one type of seller can be identified by all consumers.

In the absence of good reviews, sophisticated consumers always stop purchasing the product in finite time, so they always reveal a low-type seller. Therefore, they can reveal a high-type seller if and only if naive consumers stay in the market forever so that a good review eventually arrives. In this case the naive consumers also reveal a high-type seller. This happens only if sufficiently many bad reviews are deleted (which is the case in all equilibria if $\mu < 1/2$). Conversely, if sufficiently many bad reviews are revealed in equilibrium ($\mathcal{R}$ is sufficiently dense), then absent a good review, naive consumers also stop buying the product almost surely as $t \to +\infty$. In this case they reveal a low-type seller, but neither group of consumers is guaranteed to reveal a high-type seller.

[63] Correct decision is purchasing the product if and only if it is of high quality. Asymptotic mislearning is related to herding on suboptimal alternatives, see Banerjee [1992], Bikhchandani et al. [1992].

General Information Structures

Following the literature on experimentation, we have adopted the "conclusive good news" structure in our model, so that any good review is a conclusive evidence of $\theta = H$. However, it has been noted that in experimentation models some interesting results disappear with the transition to *"conclusive bad news"* case (see, e.g., Keller and Rady [2015], Halac and Kremer [2020]). In the context of our model this information structure would mean that both types of sellers can generate good reviews but only the low-type seller can receive bad reviews. The following proposition says that in this case no bad reviews are ever revealed.

Proposition 15. *In all equilibria under "conclusive bad news" $\mathcal{R} = \emptyset$.*

The intuition behind the proposition is trivial. Under conclusive bad news, any bad review reveals to all future consumers that $\theta = L$, meaning they have no reason to buy the product. Therefore, revealing any bad review is a weakly dominated strategy, and a short proof shows that it is actually strictly dominated.

Of course, possible information structures are not exhausted by the conclusive news cases. Another setting we explore below is one where both good and bad reviews are inconclusive. In particular, consider a *"general"* setting, defined as follows: the low-quality product yields utility 1 with probability q_+^L and utility 0 with probability $q_-^L = 1 - q_+^L$. The respective probabilities for the high-quality product are q_+^H and $q_-^H = 1 - q_+^H$, with $q_+^H > q_+^L$. Let $\bar{p}$ be such that

$$\bar{p}q_+^H + (1 - \bar{p})q_+^L = c.$$

Denote bad and good reviews in this setting as $l \in \{-, +\}$ respectively. Let $r_-^\theta(p)$ and $r_+^\theta(p)$ denote the probability with which seller of type θ reveals a bad review and a good review, respectively, in state p. Let $\mathcal{R}_l := \{p \mid (r_l^H(p), r_l^L(p)) \neq (0, 0)\}$ for $l \in \{-, +\}$. Let $f_l^\gamma(p)$ denote the belief of type-γ consumer who observes a review $l \in \{-, +\}$ posted in state p.

Note that even though this "general" setting is binary, for purposes of our result any setting with more than two reviews/utility levels can be reduced to this general setting by banning (i.e., setting $\mathcal{R}_l = \emptyset$ for) all but two reviews. For simplicity we also assume that $q_+^H \cdot q_-^H \geqslant q_+^L \cdot q_-^L$ (so that $f_+^n(f_-(p)) \geqslant p^n$), but this is not a vital assumption.

The following proposition says that in this setting we can still construct an equilibrium in which all revealed bad reviews (and good reviews alike) improve the seller's reputation among sophisticated consumers – and again bad reviews are revealed in a nonempty set of states in a payoff-relevant way.

Proposition 16. *If $\mu \in (1/2, 1)$, then there exists an equilibrium in the general setting such that*

1. $f_l^s(p) > p^s$ *for all* $p \in \mathcal{R}_l$ *and all* $l \in \{-, +\}$;
2. $\mathcal{R}_- \neq \emptyset$;
3. *this equilibrium is payoff-distinct from fully censored equilibrium.*

Fully censored equilibrium in this case can mean either one with $\mathcal{R}_- = \mathcal{R}_+ = \emptyset$ (i.e., one in which all good and bad reviews are censored), or one with $\mathcal{R}_- = \emptyset$ and same $\mathcal{R}_+$ as in the equilibrium under consideration. The latter definition ensures that payoff-nonequivalence is driven by differences in $\mathcal{R}_-$ and not $\mathcal{R}_+$.[64]

The equilibrium constructed in the proof is somewhat more restrictive than that in Theorem 3. In particular, the construction involves $\mathcal{R}_- = \mathcal{B}_{1+}^{\downarrow}$ and $\mathcal{R}_+ = \mathcal{B}_{-1}^{\uparrow}$ (where $\mathcal{B}_{-1} = \{(p^n, p^s) \in \mathcal{B}_0 \mid (f_-^n)^{-1}(p^n) \geqslant \bar{p}\}$). The important part is that the equilibrium constructed in the proof of Proposition 16 still exhibits relevant economic forces. In particular, now both seller types can bring naive consumers back to the market after driving them out, but this is still cheaper for the high-type seller because he faces a higher rate of arrival of good reviews. Consequently, the high type is more willing to lose naive consumers in the first place, which enables bad reviews' signaling function for sophisticated consumers.

Fake Reviews

Suppose that in addition to reviews written by consumers, the seller is able to post fake reviews of his choice. As we show below, our main result (Theorem 3) survives in this case.

Adopt the general setting presented in the previous subsection. Suppose that now the seller also receives opportunities to post any fake review he wants (good or bad) in addition to releasing consumers' real reviews. Future consumers cannot distinguish real reviews and fake reviews. Fake review opportunities arrive with some finite Poisson intensity λ_ϕ, which serves as a proxy for the cost of posting a fake review.[65] This rate λ_ϕ can be arbitrarily high.

[64] One can also compare the equilibrium constructed in Proposition 16 to the one with $\mathcal{R}_- = \emptyset$ and $\mathcal{R}_+ = [0, 1]^2$ with similar results.

[65] We interpret the low opportunity arrival rate λ_ϕ as high posting cost, which makes the seller reluctant to post fake reviews very frequently, and vice versa.

For simplicity we impose the same assumptions on fake review opportunities as we do on censorable reviews. Most importantly, opportunities are perishable: given that an opportunity has arrived in some state p, the seller has to decide whether to exercise it immediately, otherwise the opportunity vanishes. The seller also cannot delete fake reviews that he posted in the past.

Let $\phi_l^\theta(p) \in [0,1]$ denote the probability with which seller of type θ fakes review $l \in \{-,+\}$ in state p given that the opportunity. An obvious restriction is $\phi_-^\theta(p) + \phi_+^\theta(p) \leqslant 1$ for any θ, p.

A type-θ seller's strategy in the fake reviews setting is then given by $\{r_l^\theta, \phi_l^\theta\}$ for $l \in \{-,+\}$. Rational consumers' beliefs are updated as

$$\frac{f_l^s(p)}{1-f_l^s(p)} = \frac{p^s}{1-p^s} \cdot \frac{\lambda q_l^H r^H(p) + \lambda_\phi \phi_l^H(p)}{\lambda q_l^L r^L(p) + \lambda_\phi \phi_l^L(p)} \tag{16}$$

after review $l \in \{-,+\}$, and as

$$\dot{p}^s = p^s(1-p^s) \cdot \left(\lambda \sum_{l \in \{-,+\}} \left[q_l^H \left(1 - r_l^H(p)\right) - q_l^L \left(1 - r_l^L(p)\right) \right] - \lambda_\phi \sum_{l \in \{-,+\}} \left[\phi_l^H(p) - \phi_l^L(p) \right] \right) \tag{17}$$

in the absence of reviews. We assume that naive consumers ignore the possibility of fake reviews in the same way that they ignore censorship; hence their belief p^n is still frozen in the absence of reviews, and their reaction to review l is given by

$$\frac{f_l^n(p)}{1-f_l^n(p)} = \frac{p^s}{1-p^s} \cdot \frac{q_l^H}{q_l^L}. \tag{18}$$

To show that our main result survives in this setting, we take the equilibrium constructed in the proof of Proposition 16 and show that an analogous strategy profile is an equilibrium in fake reviews setting.

Proposition 17. *If $\mu \in (1/2, 1)$, then there exists an equilibrium in the general setting with fake reviews such that*

1. *$f_l^s(p) > p^s$ for all $p \in \mathcal{R}_l$ and all $l \in \{-,+\}$;*
2. *$\mathcal{R}_- \neq \emptyset$;*
3. *this equilibrium is payoff-distinct from fully censored equilibrium.*

Fully censored equilibrium here has the same possible meanings as in Proposition 16. The equilibrium constructed in the proof features $\phi_+^\theta(p) = 1$ whenever $p \in \mathcal{R}_+$, i.e., both types of seller post fake positive reviews at every opportunity. This dilutes the positive signal contained in good reviews but does not eliminate it completely: $f_+^s(p) > p^s$ but $f_+^s(p) \to p^s$ as $\lambda_\phi \to \infty$.

More interestingly, the equilibrium also features $\phi_-^H(p) = 1$ for all $p \in \mathcal{R}_-$: the high type strictly prefers to post fake negative reviews for his own product.[66] This is because, as in the baseline model, the low-type seller is always indifferent between revealing bad reviews and deleting them, while the high type extracts a strictly positive value from signaling through bad reviews (at least in $\mathcal{B}_{1+}^{\downarrow}$). The high-type seller writes fake bad reviews only so that he can impress sophisticated consumers with them. Sophisticated consumers then do indeed improve their opinion about product quality, even despite (and actually thanks to) the fact that they are fully aware that bad reviews they observe are likely fake and do not stem from any consumer's actual experience.

Conclusion

This paper demonstrates that the presence of naive consumers in the market may incentivize the seller to reveal bad reviews even in the presence of an opportunity to costlessly delete them. We show that bad reviews in this setting can be used as a signaling device by the seller with a high-quality product. Revealing bad reviews hurts sales to naive consumers, which he can regain through good reviews more easily than a seller with a low-quality product. This extra information contained in the fact that a bad review was not deleted makes sophisticated consumers perceive bad reviews more favorably than in the absence of censorship. Furthermore, this between-the-lines information outweighs the inherent negativity of the review, making sophisticated consumers improve their opinion about the product upon observing a bad review.

Important simplifying assumptions incorporated in the model include the seller's monopoly in the market and his inability to set the price freely, which are in some sense contradictory. Whether the effects demonstrated in this paper survive under competition and/or free pricing of the product is a possible direction for future work.

[66]To clarify, the two features – $\phi_+^\theta(p) = 1$ whenever $p \in \mathcal{R}_+$ and $\phi_-^H(p) = 1$ whenever $p \in \mathcal{R}_-$ – can coexist in the constructed equilibrium because $\mathcal{R}_- \cap \mathcal{R}_+ = \emptyset$.

The Limits of Social Learning[67]

joint with Egor Starkov

Introduction

Whenever information is dispersed in the society, the question of social learning arises: can the society aggregate this information and achieve an efficient outcome for its members? In recent times, online customer reviews have become a powerful tool of social learning: according to multiple surveys of internet users, at least a half of respondents use ratings and online reviews "always" or "often" to inform their purchasing decisions, and most respondents find reviews to be at least "mostly reliable" (Competition & Markets Authority [2015], Mintel [2015], eMarketer [2018]). Curiously, only about 10% of respondents to one of the aforementioned surveys say that they find product reviews "very reliable" (eMarketer [2018]). This skepticism can arise due to a variety of reasons, which mostly include various ways in which sellers can meddle with reviews, such as censorship and fake reviews.[68] However, in this paper we show that even in the absence of any intervention from sellers, reviews can get noisy organically.

To understand the source of this noise, which stems from *how* customers write reviews, one must first ask *why* customers write reviews. Surveys consistently produce a few modal answers to this question, with one of the most popular ones being "to help other consumers" (Trustpilot [2018]). Caring about other consumers making the right choice is often a sufficient incentive for people to spend their time and effort writing a review. These altruistic concerns, however, seem to only appear ex post – after the consumer has purchased and consumed the product – rather than ex ante. In particular, when choosing which product to buy, the consumers appear to focus primarily on their own expected utility from consumption, rather than on their desire to provide helpful information to others.

This inconsistency in altruism, as we show, must lead to noise in product reviews. When product quality is uncertain, purchases have an informational externality, since in addition to direct consumption utility they allow informative reviews to be written, which allow future consumers to make more efficient decisions. However, when deciding on the purchase, a self-interested consumer does not internalize this information-generating effect, and so their private expected value from

[67] This paper should be cited as A. Smirnov, E. Starkov. The Limits of Social Learning. mimeo, 2020.

[68] See, e.g., Luca and Zervas [2016] for an exploration of the effects of fake reviews and Smirnov and Starkov [2020] for a model of censorship in product reviews.

buying a product is always lower than social value. This discrepancy is, in turn, recognized by an altruistic reviewer, who may in some circumstances want to mislead a future consumer into buying a product when it is not individually optimal to do so.

We formalize the argument above in a model of product reviews, in which a sequence of consumers decide whether to buy a product of some uncertain quality and, if they do, what kind of review to write about their experience. A consumer in our model only purchases the product if her expected consumption utility warrants this. The realized utility is informative about the product quality. The consumer can leave a review describing her consumption experience, and when doing so she wishes to maximize welfare of consumers that arrive at the market after her.

The myopic behavior at the purchasing stage and the altruistic desire to induce some experimentation with the product at the reviewing stage conflict with each other. We show that this conflict creates noise in communication through reviews. Instead of reporting their experiences truthfully, the consumers obfuscate their reviews to foster experimentation. This is true regardless of whether consumers can commit to some communication strategy (which should be interpreted as a shared social norm among consumers) or not. In the latter case, every consumer leaves a review they believe to be socially optimal given their experience. This scenario produces even more interesting results.

In particular, we show that if a consumer cannot commit to a communication strategy, then despite the conflict arising only in a special set of circumstances – when the product is believed to be good enough to experiment with socially, but not good enough to buy just for the sake of doing so, – we show that the effects of this conflict propagate and distort communication in other cases as well. More specifically, communication must then take the interval structure known in the cheap talk literature, when senders with similar private beliefs pool on the same message.

Two conclusions may be drawn from our results. Firstly, coarse categories in product reviews (such as one- to five-star ratings) are almost sufficient for information transmission in the presence of the aforementioned experimentation conflict. Allowing free-form reviews in addition to – or instead of – such ratings will not significantly increase the amount of information available to future consumers (unless the original categories were too coarse). Secondly, our paper provides a possible explanation for inflation in product reviews, namely that reviews are inflated in order to deceive future consumers into purchasing the product they would not have bought otherwise. This complements other possible explanations, including positive ratings being sponsored or just fake. Contrary to those explanations, in our case inflation arises endogenously as a result of interaction between consumers, with no intervention from the firm whatsoever.

This paper contributes to the social learning literature. A lot of the existing literature has focused on non-strategic learning in local settings, such as networks. This is driven by recognition that we as consumers receive a lot of our information second-hand, so it may be distorted by other agents' perceptions and beliefs. In turn, the part of the literature that deals with *strategic* learning has mostly focused on strategic information *acquisition*, forcing the agent with a limited learning capacity to choose their information sources carefully. Our paper focuses instead on social learning with strategic information *provision*. For a detailed literature review, see below.

The paper is organized as follows. Section II contains a review of the relevant literature. In Section II we formulate the general version of the model. Section II presents the main result – that no perfect communication is possible in our setting – and explains the intuition behind it. Sections II and II then discuss what communication structures *can* arise in our model, with Section II exploring an illustrative three-period example, and Section II generalizing the insights to an infinite-horizon problem. Section II concludes. All proofs are relegated to the Appendix.

Literature Review

The current paper mainly contributes to two strands of literature: *social learning* and *dynamic cheap talk*.

The literature on social learning is vast. Our paper is closest to the literature on herding and cascades in sequential learning (Banerjee [1992], Bikhchandani, Hirshleifer, and Welch [1992], Smith and Sørensen [2000]). In these models the agents choose actions which are payoff-relevant for agents themselves and, at the same time, signal their private information to subsequent agents. Smith and Sørensen [2011] provide an excellent overview of the topic. The most recent general treatment of the setting is provided by Xu [2018]. Most relevant are works by Ali and Kartik [2012] and Smith, Sørensen, and Tian [2017], who consider sequential observational learning with others-regarding preferences. However, in the observational learning framework agents receive private information and act on it; actions are the only source of information for future agents who observe neither past signals, nor past outcomes, while our paper explores learning under strategic communication. Wolitzky [2018], in contrast, considers sequential learning when players observe outcomes of previous players, but not their actions. This is closer to our paper, except outcomes in our model are observed through noisy communication rather than directly. Ali [2018] studies observational learning with costly information acquisition.

Social learning with strategic information provision was explored by Swank and Visser [2015] when the conflict arises from senders' career concerns. Liang and Mu [2019] consider a model where agents can, similarly to our paper, be tempted by exploiting myopic benefits which prevents future generations from learning the state correctly. Au [2019] presents a model of (non-social) learning, in which experts' recommendations to the agent are distorted even despite the seeming absence of conflict between the parties.

A separate strand of the social learning literature focuses exclusively on learning from customer reviews (e.g., Acemoglu, Makhdoumi, Malekian, and Ozdaglar [2017] and Vaccari, Maglaras, and Scarsini [2018]).

The decentralized literature of decision-making and communication in our model relates us to literature on social learning on networks, which are inherently decentralized. Lobel and Sadler [2015] and Arieli and Mueller-Frank [2019] study sequential social learning when agents are arranged in a network or into an m-dimensional integer lattice, respectively. Campbell [2013] explores pricing and advertising in networks of friends who learn via word-of-mouth communication. Galeotti, Ghiglino, and Squintani [2013], Schopohl [2017] and Foerster [2019] analyze various games of strategic information transmission in networks. Migrow [2018] studies how a manager should design a communication network in an organization to optimally elicit the information from employees. The conflicts explored in these papers are different from what we focus on in this paper.

Literature on the *design* of social learning considers the information structures that incentivize short-lived agent to experiment for the sake of society. Notable references include Kremer, Mansour, and Perry [2014], Che and Hörner [2018], Mansour, Slivkins, and Syrgkanis [2019], and Cohen and Mansour [2019].[69] We explore what effectively is a decentralized version of these models, with each agent trying to communicate in a way so as to create optimal experimentation incentives, but lacking the commitment power and memory of a single principal.[70]

Our paper models communication via cheap talk a là Crawford and Sobel [1982]. Other models (apart from ours) of sequential communication include Ambrus, Azevedo, and Kamada [2013], Renault, Solan, and Vieille [2017], and Chiba [2018]. Le Quement and Patel [2018] explore cheap talk with preferences for reciprocity.

Our model presents consumers as altruistic when they are writing reviews. It has been argued for a long time that the economic model of homo economicus as a self-interested agent does not fully

[69] Optimal experimentation by a group of long-lived agents with incentives for free-riding was studied by Bonatti and Hörner [2017], Keller, Rady, and Cripps [2005], and Hörner, Klein, and Rady [2015]. Heidhues, Rady, and Strack [2015] move from observable to private payoffs and explore communication in this setting.

[70] We also consider the case when consumers can commit to a specific message structure.

capture real-world behavior, which often exhibits regard for others. Various classical explanations are Andreoni [1990] (impure altruism), Fehr and Schmidt [1999] (inequality aversion) and Becker [1974] (pure altruism). The literature is surveyed in Fehr and Schmidt [2003], Konow [2003], and Meier [2006]. More recently, an attempt to provide an axiomatic foundation for such preferences has been made by Galperti and Strulovici [2017].

Within the context of social learning, experiments by March and Ziegelmeyer [2016] and Peng, Rao, Sun, and Xiao [2017] find evidence of altruistic motives when testing standard models of observational learning.

The Model

Primitives

Time is discrete and infinite: $t \in \{1, 2, \ldots, \}$. All agents share a common discount factor $\beta < 1$.

Seller. There is a single long-lived seller, who offers for sale a single product that he has in infinite supply. Product quality θ, which represents the average consumption utility of the product, can be either *low* or *high*: $\theta \in \{L, H\}$, with $0 \leqslant L < H$. The price of the product is fixed at $c > 0$; to avoid triviality we assume that $L < c < H$.

Consumers. Each period a single short-lived risk-neutral consumer arrives at the market. The consumer can either purchase the good at cost c or leave the market forever, receiving the reservation utility normalized to 0. In case of purchase, the consumer receives random consumption utility s, distributed according to quality-contingent cdf F^θ with mean θ and respective pdf f^θ. We assume that both F^L and F^H have full support on the same open interval $S = (\underline{s}, \overline{s}) \subseteq \mathbb{R}$.[71] Both measures are absolutely continuous on S, and their respective densities are continuously differentiable and bounded from above. In addition, we assume that MLRP holds:

Assumption 2 (MLRP)**.** *Ratio* $\frac{f^H(s)}{f^L(s)}$ *is a strictly increasing and continuous function of* s *on* S*. Moreover,* $\lim\limits_{s \to \underline{s}} \frac{f^H(s)}{f^L(s)} = 0$*, and* $\lim\limits_{s \to \overline{s}} \frac{f^H(s)}{f^L(s)} = +\infty$.[72]

The consumer does not observe product quality θ, so her purchasing decision is based on her belief $p = \mathbb{P}(\theta = H)$. In particular, the consumer purchases the product if and only if her expected

[71] Here $\underline{s} = -\infty$ and $\overline{s} = +\infty$ are both admissible values.

[72] Also note that MLRP implies that F^H first order stochastically dominates F^L.

consumption utility exceeds the cost of purchase:

$$\theta(p) := Hp + L(1-p) \geqslant c \ \Leftrightarrow \ p \geqslant \bar{p},$$

where $\bar{p} := \frac{c-L}{H-L}$.[73] This purchasing strategy will be taken as given in what follows.

Reviews. If the good was purchased, the consumer then sends a cheap talk message $m \in \mathcal{M}$ (writes a review) to subsequent consumers, describing her experience with the product. The message set $\mathcal{M}$ is assumed to be arbitrarily rich, with $[0,1] \subseteq \mathcal{M}$. When leaving a review, the consumer maximizes the expected discounted sum of consumption utilities of all future consumers.

We consider two regimes. Under the *commitment regime* a consumer can commit to some utility-contingent reporting strategy $(\underline{s}, \overline{s}) \to \mathcal{M}$ *before* a purchase. The interpretation of this regime is that there exists a welfare-maximizing social norm, which prescribes the mapping from experiences to reviews. Under the *no commitment regime*, the consumer chooses m *after* observing her consumption utility s. The latter regime is also referred to as the decentralized scenario.

Timing. Within a given period, the order of events is as follows:

1. Time-t consumer arrives at the market and observes all past reviews $(m_1, m_2, \ldots, m_{t-1})$ and forms belief p_t about the quality of the product.

2. The consumer decides whether to purchase the product at cost c or not.

3. After a purchase she receives random consumption utility $s_t \sim F^\theta$ and updates her belief about the product quality.

4. After a purchase the consumer leaves review m_t about her experience observable to all subsequent consumers. A consumer who has not purchased the product leaves no review: $m_t = \varnothing$.[74]

Histories and State Variables

Review history $R_t := (m_1, m_2, \ldots, m_{t-1})$ is a tuple consisting of all messages sent by consumers before period t. It constitutes the public history at the beginning of period t. We denote the *public*

[73]We assume that the consumer purchases the product when indifferent.

[74]For simplicity, we assume that $\varnothing \notin \mathcal{M}$, i.e., a purchasing consumer cannot stay silent and must leave a meaningful review.

belief about the quality of the product as $p_t := \mathbb{P}(\theta = H \mid R_t)$. The prior $p_1 = \mathbb{P}(\theta = H|\varnothing)$ is exogenously fixed and commonly agreed upon.

The *private posterior belief* of time-t consumer in case she purchased and consumed the product is denoted by $b_t := \mathbb{P}(\theta = H \mid R_t, s_t)$. Given p_t and s_t we can compute b_t as

$$b_t = \frac{p_t f^H(s_t)}{p_t f^H(s_t) + (1 - p_t)\, f^L(s_t)}. \tag{19}$$

Let $\mu^\theta(b_t \mid p_t)$ denote the cdf of a distribution of b_t induced by s_t conditional on p_t and true state θ.[75]

The belief p_t contains all payoff-relevant information available to time-t consumer at the time she decides whether to purchase the product. The pair of beliefs p_t and b_t summarizes all payoff-relevant information available to time-t consumer when she decides which message to send to subsequent consumers. In what follows, we will focus on a setting, in which we treat belief p_t and current time t as a sufficient statistic of the review history R_t.[76] Because of this, we call the tuple (p_t, b_t, t) the *private state* of time-t consumer, and we refer to (p_t, t) as the time-t *public state*. We will typically omit t from the description of states, given that it can be inferred from belief indexing.

Given that the consumers' purchasing decisions are myopic and described by "buy iff $p_t \geqslant \bar{p}$", from this point onward we will be focusing on consumers' *communication* strategies. The time-t consumer's behavioral strategy is r, where $r(m|p_t, b_t)$ is the probability with which the time-t consumer sends message $m \in \mathcal{M}$ in private state (p_t, b_t). Let $\mathcal{M}(p_t) = \{m \in \mathcal{M} \mid \exists\, b_t : r(m \mid p_t, b_t) > 0\}$. Then the public belief p_{t+1} induced by message $m \in \mathcal{M}(p_t)$ is given by

$$p_{t+1} = q(p_t, m) := \frac{p_t \cdot \int_0^1 r(m|p_t, b_t) d\mu^H(b_t|p_t)}{p_t \cdot \int_0^1 r(m|p_t, b_t) d\mu^H(b_t|p_t) + (1 - p_t) \cdot \int_0^1 r(m|p_t, b_t) d\mu^L(b_t|p_t)}. \tag{20}$$

We let $\mathcal{P}(p_t) = \{q(p_t, m) \mid m \in \mathcal{M}(p_t)\}$ denote the set of all posteriors which are induced by time-t consumer in equilibrium. We partition this set into $\mathcal{S}(p_t) \cup \mathcal{E}(p_t) = \mathcal{P}(p_t)$. Here $\mathcal{E}(p_t) = \{q \in \mathcal{P}(p_t) \mid q \geqslant \bar{p}\}$ includes all posteriors for which the next consumer purchases the product, while $\mathcal{S}(p_t) = \{q \in \mathcal{P}(p_t) \mid q < \bar{p}\}$ contains all posteriors which deter the next consumer

[75] It can be computed explicitly: $\mu^\theta(b_t \mid p_t) = F^\theta\left(l^{-1}\left(\ln\left(\frac{b_t}{1-b_t}\right) - \ln\left(\frac{p_t}{1-p_t}\right)\right)\right)$, where l^{-1} is an inverse function to $\ln\left[\frac{f^H(s)}{f^L(s)}\right]$.

[76] This is not without loss: if two time-t review histories produce the same p_t, they will be treated as equivalent. This would preclude the possibility of having different continuation equilibria after the two histories.

from the purchase. Note that if $p_t \geqslant \bar{p}$ then $\mathcal{E}(p_t) \neq \emptyset$, as the public belief p_t is a martingale. Further, note that if $p_t < \bar{p}$ then the market shuts down: time-t consumer does not purchase the product, does not write a review, hence at $t+1$ the next consumer has exactly the same information at the time she makes her purchasing decision (i.e., $p_{t+1} = p_t$) and does not purchase the product either. Therefore, all $q \in \mathcal{S}(p_t)$ are equivalent in the sense of shutting the market down. Hereinafter we will without loss only consider a representative element of $\mathcal{S}(p_t)$ whenever it is nonempty.

Maximization Problem

When a consumer sends message m at private state (p_t, b_t) and induces public belief $p_{t+1} = q$, her value (the discounted sum of future consumers' utilities) from doing so is equal to

$$V(q \mid p_t, b_t) := \mathbb{E}\left[\sum_{j=t+1}^{+\infty} \beta^{j-t-1} \cdot \mathbb{I}\left(p_j \geqslant \bar{p}\right) \cdot s_j \,\middle|\, p_{t+1} = q\right]. \tag{21}$$

The expectation is taken over all future histories that start with public belief $p_{t+1} = q$. Implicit in (21) is the correlation between future s_j and future p_j stemming from future consumers' equilibrium strategies. Maximizing (21) over all available messages, we get the consumer's optimal value in private state (p_t, b_t):

$$V(p_t, b_t) = \max_{p \in \mathcal{P}(p_t)} V(p \mid p_t, b_t).^{77}$$

For a given equilibrium, the time-t consumer's *ex ante* expected continuation value conditional on public belief p_t is given by

$$V(p_t) := \mathbb{E}\left[V(p_t, b_t)\right] = p_t \cdot \int_0^1 V(p_t, b_t) d\mu^H(b_t|p_t) + (1-p_t) \cdot \int_0^1 V(p_t, b_t) d\mu^L(b_t|p_t).$$

When talking about values, we will use superscripts C or D to distinguish commitment and no-commitment (decentralized) solutions.

Equilibrium Definition

We are looking for Perfect Bayesian Equilibria of the game, which consist of a strategy profile $r(m|p_t, b_t)$ and updating rules for beliefs p_t and b_t such that

[77] This representation implies that the consumer chooses a message from $\mathcal{M}(p_t)$ rather than $\mathcal{M}$. This is a simplifying assumption: we do not allow to send out-of-equilibrium messages so that we do not have to keep track of beliefs after such messages. This restriction is without loss.

Belief Consistency: (19) holds at all private histories (p_t, b_t), and (20) holds after all p_t and $m \in \mathcal{M}(p_t)$;

C-Optimality: For commitment regime: $r(m \mid p_t, b_t)$ is chosen so as to maximize $V^C(p_t)$ for all p_t;

D-Optimality: For decentralized regime: if $m \in \mathcal{M}(p_t)$ and $r(m \mid p_t, b_t) > 0$ then $V^D(p_t, b_t) = V^D(q(p_t, m) \mid p_t, b_t)$.

Belief consistency condition ensures that consumers use Bayes' rule whenever possible to update their belief. C-Optimality states that in the commitment scenario, the consumer chooses a mapping from private belief b_t to messages (conditional on p_t and subject to Belief Consistency) so as to maximize the ex ante value. In the decentralized game, D-Optimality requires that the consumer maximizes her ex post value (after learning s_t).

No Perfect Communication

This section demonstrates the main idea of this paper: that truth-telling is neither an optimal social "norm" for writing reviews (i.e., it's not a commitment solution), nor it is an equilibrium in the decentralized market. This conclusion is driven by the implicitly lexicographic nature of consumers' preferences. When buying the product, a consumer maximizes own expected utility, but when writing a review, she cares about all future generations. The consumer can thus be represented as having lexicographic preferences: the first-order preference is for own well-being, while the warm glow from social welfare is second-order. The consumer is thus unwilling to sacrifice her consumption utility for sake of the society. This creates a conflict, since time-t consumer would like the consumer at $t + 1$ to conduct socially efficient experimentation, possibly by buying the product even when it is not myopically optimal, so that more information about product quality is generated. This conflict introduces noise into communication between the two generations of consumers, i.e., into the review of time-t consumer.

To formulate the result, we first need to introduce the notion of a cascade. Cascades are prominent in the observational learning literature, where this label is used whenever the society gets locked into one of the available alternatives (possibly at a loss to efficiency). We use it in the same context.

Definition 8. *Message $m \in \mathcal{E}(p(R_t))$ at public history R_t starts a* cascade *if $p(R_s) \geqslant \bar{p}$ for all $R_s = (R_t, m, \ldots)$.*

In other words, we say that a recommendation to purchase issued at R_t leads to all future consumers buying the product, regardless of any of the interim consumers' experiences and reviews. Once a cascade starts, no new reviews can change the consumers' behavior. There are two things to note in relation to cascades. First, any message $m \in \mathcal{S}(p_t)$ starts a cascade as well, in the sense that no future consumers buy the product again, as discussed in Section II.[78] Second, in the no-commitment scenario there always exists a continuation equilibrium in which any given $m \in \mathcal{E}(p(R_t))$ starts a cascade. One example is the babbling equilibrium, one in which all future reviews are uninformative and are perceived as such, and thus the public belief remains frozen at $p(R_t, m)$.[79] However, in general, a cascade need not shut down the information transmission completely: reviews may be informative and affect the public belief p_t as long as they do not affect future consumers' actual purchasing decisions.

The following proposition presents the main result of this section, which motivates further discussion. This result shows that truth-telling (fully revealing communication) is neither a welfare-maximizing social norm, nor it is an equilibrium in a decentralized game.

Proposition 18. *In the commitment regime,* $\mathcal{P}(p_t) = [0, 1]$ *for all* p_t *does not deliver a maximum for* $V^C(p_t)$.

In the no-commitment regime, $\mathcal{P}(p_t) = [0, 1]$ *if and only if any message available at* p_t *starts a cascade.*

Proposition 18 shows that the conflict between the sender and the receiver of a review precludes perfect communication. If the sender's (time-t consumer's) posterior b_t is just below the myopic cutoff $\bar{p}$, she generally wants the next consumer to purchase the product and generate information about quality for the sake of future generations. The receiver (consumer at $t + 1$), however, would not buy the product if she learned that given all available information, the product is good only with probability $b_t < \bar{p}$. The sender thus wants to misrepresent her posterior as if it was barely above $\bar{p}$. As we show in the following sections, in the absence of commitment this noisiness of communication unravels – even though the sender-receiver conflict only exists for some $b_t < \bar{p}$ the noise propagates to all $b_t > \bar{p}$.

The statement for the no-commitment regime also illustrates that the noise arises exactly from the sender's regard for consumers beyond time $t+1$. In particular, if no informative communication

[78] The definition above relates to positive cascades, while $m \in \mathcal{S}(p_t)$ starts a negative cascade.

[79] Babbling is prominent in cheap talk literature; to see that it is an equilibrium note that neither player has a profitable deviation. The sender cannot benefit by sending informative messages because they are ignored by the receivers regardless, and the receivers cannot benefit by following the sender's recommendation since it is uninformative.

is possible at $t+1$ or afterwards, then time-t consumer has no reason to induce experimentation that the consumer at $t+1$ is trying to avoid, because the information from these experiments would not be conveyed to the subsequent generations either way.

In case of commitment, the reasoning behind the result is more involved. On the one hand, it is still desirable for the sender to pool the states just below $\bar{p}$ with those just above it to induce more experimentation at $t+1$. On the other hand, however, distorting communication of states $b_t > \bar{p}$ is costly, since this is not only concealing some information about the state from the future consumers, but it may also decrease the amount of experimentation from $t+2$ onwards. The latter statement holds because pooling depresses beliefs p_{t+1} induced after $b_t > \bar{p}$, as compared to truth-telling. However, the gains from pooling states $b_t \in (\bar{p}-\varepsilon, \bar{p}+\varepsilon)$ are approximately equal to $V^C(\bar{p}) \cdot \varepsilon > 0$, while the losses are of order $\mathcal{O}(\varepsilon^2)$ because $V^C(p_t)$ is continuous in p_t. Truth-telling is thus not optimal under commitment either. A more detailed exposition of this logic is presented in Subsection II.

In this section we explored how equilibria cannot look. The following sections provide more insight into how they do look, with and without commitment. We begin with exploring the most basic version of the model.

Three-Period Example

This section demonstrates the main insights in the most basic three-period setting. For sake of this example, assume that consumption utilities s_t are normally distributed with mean θ and variance σ^2. Suppose further that there is no discounting. We shall denote the three consumers as C1, C2, and C3 respectively. We solve the example by backward induction.

C3 purchases the product if and only if $\theta(p_3) = Hp_3 + L(1-p_3) \geqslant \bar{p}$, and her messaging strategy is irrelevant, since no consumers arrive at the market after her.

Second Period

If $p_2 < \bar{p}$ then, as mentioned in the model setup, the game effectively ends: C2 does not buy the product, so writes no review, so $p_3 = p_2 < \bar{p}$, and C3 does not buy the product either. All payoffs starting from $t = 2$ are zero in this case. Conversely, if $p_2 \geqslant \bar{p}$ then C2's continuation value equals C3's expected consumption utility: $V(p_3|p_2, b_2) = \theta(p_3) - c$. Therefore, there is no conflict between C2 and C3, and truthful communication, where C2 reports $m_2 = p_2$, is possible in equilibrium (in both regimes – with and without commitment).

Note, however, that the only information relevant to C3 is whether to buy the product or not. She cannot make use of more precise information to make better recommendations to future consumers because there are no future consumers. Therefore, all continuation equilibria from p_2 with $\mathcal{S}(p_2) \neq \emptyset$ are payoff-equivalent to the one where only two messages are used: $\mathcal{M} = \{$"buy", "do not buy"$\}$. In this case "buy" is sent by C2 whenever $b_2 \geqslant \bar{p}$, and "do not buy" is sent when $b_2 < \bar{p}$. Then

$$V(p_2, b_2) = \max\{\theta(b_2) - c, 0\}.$$

If $\mathcal{S}(p_2) = \emptyset$ then any message at p_2 starts a cascade, so $V(p_2, b_2) = \theta(b_2) - c$ in this case.

First Period

In this section we analyze $V(p_2|p_1, b_1)$, C1's continuation value from inducing prior belief p_2 for C2, when her own private belief is b_1. We again look at states $p_1 \geqslant \bar{p}$ (otherwise all values are zero). For simplicity we assume that all time-2 continuation equilibria are informative, i.e., $\mathcal{S}(p_2) \neq \emptyset$ for all $p_2 \geqslant \bar{p}$.[80] Note that since truthtelling is both a first-best and an equilibrium outcome at $t = 2$ – and any informative continuation is equivalent to truth-telling, as shown above – the value $V(p_2 \mid p_1, b_1)$ for given p_2 and b_1 is the same in both commitment and no-commitment regimes. The difference between the two only lies in what communication strategies can be optimal for C1 given $V(p_2|p_1, b_1)$. These strategies are explored in sections II and II.

With $p_1 \geqslant \bar{p}$, C1 buys the good and receives utility s_1. If she sends a message $m \in \mathcal{S}(p_1)$ that induces no further purchases, her continuation value equals zero. When she sends $m \in \mathcal{E}(p_1)$, C2 purchases the product and obtains utility s_2. Following that, C3 purchases the product if and only if $p_3 = b_2(s_2) \geqslant \bar{p}$. Signal $\bar{s}_2$ which induces $b_2 = \bar{p}$ can be found from

$$\frac{b_2}{1 - b_2} \equiv \frac{p_2}{1 - p_2} \cdot \frac{f^H(\bar{s}_2)}{f^L(\bar{s}_2)} = \frac{\bar{p}}{1 - \bar{p}}.$$

Given that $\frac{f^H(\bar{s}_2)}{f^L(\bar{s}_2)} = e^{\frac{H-L}{\sigma^2}\left(\bar{s}_2 - \frac{H+L}{2}\right)}$, we have

$$\bar{s}_2(p_2) = \frac{\sigma^2}{H - L}\left[\ln\left(\frac{\bar{p}}{1 - \bar{p}}\right) - \ln\left(\frac{p_2}{1 - p_2}\right)\right] + \frac{H + L}{2}. \tag{22}$$

[80]The case when a cascade is started at $t = 2$ (after any message at $t = 1$) is trivial, since then truthtelling is an equilibrium by the same argument as in the second period. The case when a cascade is started by some but not all messages $m \in \mathcal{E}(p_1)$ is non-trivial, but we do not deem it worthy of careful consideration.

Therefore, if C2 buys the product, C1's continuation value from inducing some belief p_2 is given by

$$V^*(p_2 \mid p_1, b_1) = \mathbb{E}\left[s_2 + s_3 \cdot \mathbb{I}\{s_2 \geqslant \bar{s}_2(p_2)\} \mid b_1\right]. \tag{23}$$

Given C2's sequential rationality, C1's continuation value is

$$V(p_2 \mid p_1, b_1) = \begin{cases} V^*(p_2 \mid p_1, b_1) & \text{if } p_2 \geqslant \bar{p}, \\ 0 & \text{if } p_2 < \bar{p} \end{cases}$$

From the point of view of C1, the good is of high quality with probability b_1. In that case C3 buys the good with probability $1 - F^H(\bar{s}_2)$ and receives $H - c$ in expectation. Similarly, with probability $1 - b_1$ the good is of low quality, and then C3 gets $L - c$ conditional on purchase which occurs with probability $1 - F^L(\bar{s}_2)$. In the end, expression (23) can be rewritten as

$$V^*(p_2 \mid p_1, b_1) = \theta(b_1) - c + b_1 \cdot \left(1 - F^H(\bar{s}_2(p_2))\right)(H - c) + (1 - b_1) \cdot \left(1 - F^L(\bar{s}_2(p_2))\right)(L - c), \tag{24}$$

where $\bar{s}_2$ is given by (22).

Analyzing (24), we can identify several important properties of $V^*(p_2 \mid p_1, b_1)$. Firstly, it is strictly positive at $b_1 = \bar{p}$ for all p_2. This follows from the fact that $F^H(\bar{s}_2(p_2)) < F^L(\bar{s}_2(p_2))$. The function is continuous in b_1, hence it is also strictly positive in some neighborhood of $b_1 = \bar{p}$. This implies that C1 strictly prefers to induce $p_2 \geqslant \bar{p}$ for at least some values of $b_1 < \bar{p}$: she wants C2 to purchase the product despite believing that this is not myopically optimal. This is due to the social value of experimentation (i.e., of information generated by the purchase at $t = 2$), which is internalized by C1 in her review strategy, but not by C2 in her purchasing strategy. There is thus a conflict between the two.

Secondly, the expression in (24) is strictly increasing in p_2 on $[0, b_1]$ and is strictly decreasing on $[b_1, 1]$, i.e., it is single-peaked with a peak at $p_2 = b_1$. This means that when $b_1 \geqslant \bar{p}$, C1 would prefer to tell the truth to C2 and induce the correct belief $p_2 = b_1$. To see this, observe that

$$\frac{\partial V(p_2 \mid p_1, b_1)}{\partial p_2} = (1 - b_1)\,\bar{p} f^L(\bar{s}_2(p_2)) \cdot \frac{\sigma^2}{(1 - \bar{p})p_2(1 - p_2)} \left(\frac{b_1}{1 - b_1} \cdot \frac{1 - \bar{p}}{\bar{p}} \cdot \frac{f^H(\bar{s}_2(p_2))}{f^L(\bar{s}_2(p_2))} - 1\right).$$

Since $\frac{f^H(\bar{s}_2(p_2))}{f^L(\bar{s}_2(p_2))}$ is strictly decreasing in p_2 from $+\infty$ to 0, and the fraction multiplying the bracket is positive, we get that $V(p_2 \mid p_1, b_1)$ is single-peaked. We can find the peak by equating $\frac{\partial V(p_2|p_1,b_1)}{\partial p_2}$

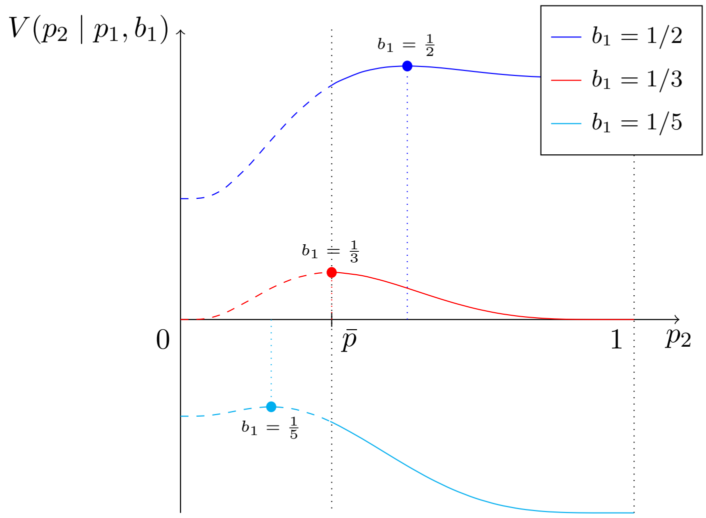

Figure 7: $V^*(p_2 \mid p_1, b_1)$ as a function of p_2.
Note: the parameter values are $H = 3, L = 0, c = 1, \sigma = 4,\ b_1 = \frac{1}{5}, \frac{1}{3}, \frac{1}{2}$.

to zero, which yields

$$\bar{s}_2 = \frac{\sigma^2}{H - L}\left[\ln\left(\frac{\bar{p}}{1 - \bar{p}}\right) - \ln\left(\frac{b_1}{1 - b_1}\right)\right] + \frac{H + L}{2},$$

which together with (22) gives condition $p_2 = b_1$.

Expression in (24) as a function of p_2 for different values of b_1 is plotted in Figure 7. Since (24) only coincides with $V(p_2|p_1, b_1)$ for $p_2 \geqslant \bar{p}$ (and $V(p_2|p_1, b_1) = 0$ otherwise), we use dashed lines for values $p_2 < \bar{p}$.

The two properties of $V^*(p_2 \mid p_1, b_1)$ outlined above – that it is positive for $b_1 = \bar{p} - \varepsilon$ for at least some $\varepsilon > 0$, and that it peaks at $p_2 = b_1$ – will be used heavily in the analysis that follows.

Interval Structure of Equilibrium Communication

We now show that in the no commitment regime, communication in the first period must have interval structure. In other words, there exists a partition $0 = \Delta_0 \leqslant \Delta_1 < \Delta_2 < \ldots = 1$ and messages $m_1, m_2, \ldots$ such that if $b_1 \in (\Delta_{j-1}, \Delta_j)$ then $r_{m_j}(p_1, b_1) = 1$.

First note that if $\mathcal{S}(p_1)$ is nonempty – i.e., if there is a review that will prevent C2 from buying the product, – this review will be used by C1 after at least some b_1 low enough.[81] Assume that this is the case in what follows.

Consider now the smallest posterior belief among those available in equilibrium that leads C2 to purchase the product, $e_1 = \min \mathcal{E}(p_1)$. Let Δ_1 denote the level of posterior belief b_1 with which

[81] For $b_1 \approx 0$, expression (24) reduces to $V^*(p_2 \mid p_1, b_1) = (L - c) \cdot \big(1 + 1 - F^L(\bar{s}_2(p_2))\big)$, which is negative because $L < c$.

C1 is indifferent between leaving a review in $\mathcal{S}(p_1)$ and review e_1. From the fact that $V^*(p_2 \mid p_1, b_1)$ is positive at $b_1 = \bar{p}$ it is immediate that $\Delta_1 < e_1$. The fact that $V^*(p_2 \mid p_1, b_1)$ has a peak at $p_2 = b_1$ implies, in turn, that all types b_1 of C1 also prefer to leave review e_1 rather than any review $p_2 > e_1$.[82]

In other words, if there exists a way to make the "most cautious recommendation to buy", then C1 would like to adopt that phrasing for a wide range of posteriors b_1. This is because she wants C2 to purchase the product, thus generating information, even when it is not myopically optimal for C2 – but does not want to distort the information that C2 passes onwards. These two goals conflict with each other, since C1 only has one stone – her review – to hit both birds.

Recall, however, that C2 is rational and Bayesian – in particular, when forming her belief p_2 she takes C1's incentives into account. Therefore, it must be the case that the prior belief $p_2 = e_1$ of C2 must incorporate the information contained in the posteriors b_1 of the versions of C1 who write reviews that induce e_1. We have argued above that there are types $b_1 < e_1$ that induce $p_2 = e_1$, so there must also be types $b_1 > e_1$ that do the same. Consider the supremum of such types b_1 and denote it by Δ_2. C1 with posterior $b_1 = \Delta_2$ must (by continuity of V^*) be indifferent between inducing e_1 and the next-lowest available posterior e_2. However, we know that $V^*(p_2 \mid p_1, b_1)$ is single peaked in p_2 with a peak at $p_2 = b_1$, hence the indifference condition $V^*(e_1 \mid p_1, \Delta_2) = V^*(e_2 \mid p_1, \Delta_2)$ implies that $e_2 > \varDelta_2$. By iterating the argument, we get that

$$... < \varDelta_j < e_j < \varDelta_{j+1} < e_{j+1} < ...$$

Plainly speaking, the fact that the aforementioned "most cautious recommendation to buy" is noisy and not perfectly revealing of b_1 implies eventually that all other messages must be noisy as well. Notably, perfect communication is thus impossible even for high posteriors b_1, when there is no conflict between the sender and the receiver.

Figure 8 plots the continuation payoff of C1 in an interval equilibrium with three messages: $e_1 = \bar{p}$, e_2, and "stop experimentation". This payoff coincides with the unconstrained maximum (when C1 can choose any p_2 and force C2 to purchase the item) whenever $b_1 = e_1$, but is strictly lower for all other posteriors. The noise in communication thus hurts C1 by making the purchasing decision of the *third* consumer less efficient, but this is compensated by the more efficient purchasing decision of C2 as compared to the case when C1 can choose any p_2 but cannot force C2 to buy.

[82] This also implies that $\mathcal{E}(p_1)$ must be closed at the bottom in any equilibrium, i.e., e_1 does actually exist.

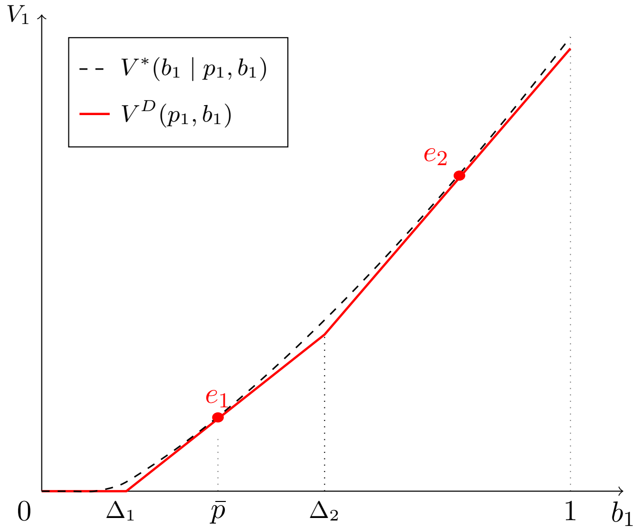

Figure 8: $V^*(b_1 \mid p_1, b_1)$ and value $V^D(p_1, b_1)$ in an interval equilibrium.
Note: $H = 3, L = 0, c = 1, \sigma = 2$. To illustrate convexity, it is also assumed for this graph that C1 cares about third consumer's utility 9 times as much as about C2's (one can think that 9 consumers arrive in the third period).

Commitment Solution

By committing to truthful communication, C1 can achieve value

$$V(b_1 \mid p_1, b_1) = \begin{cases} V^*(b_1 \mid p_1, b_1) & \text{if } b_1 \geqslant \bar{p}, \\ 0 & \text{if } b_1 < \bar{p} \end{cases}$$

Our goal in this section is to demonstrate that C1 can do better than this (in expectation over b_1 for a given p_1). The source of improvement lies in the discontinuity of $V(b_1 \mid p_1, b_1)$ at $b_1 = \bar{p}$.

To evaluate the trade-offs introduced by imperfect communication of posteriors b_1, it is useful to understand how $V(p_2 \mid p_1, b_1)$ depends on the induced prior p_2 for a given posterior b_1. To do this, we ask the dual question and visualize the dependence of $V(p_2 \mid p_1, b_1)$ on b_1 for a given p_2. In particular, (24) is a linear function of b_1, meaning that once we fix p_1 and p_2, value $V(p_2 \mid p_1, b_1)$ as a function of b_1 is given by a tangent to the maximal value $V^*(b_1 \mid p_1, b_1)$ at $b_1 = p_2$. This means that $V^*(b_1 \mid p_1, b_1)$, as well as $V(p_1, b_1)$ in any equilibrium, are convex in b_1, since all of these are upper envelopes of respective sets of linear functions.

This convexity implies that C1 cannot benefit from sending messages that pool different posteriors b_1 above $\bar{p}$. On the other hand, it is also strictly optimal to stop experimentation for all private beliefs below $\bar{\bar{p}}$, where $\bar{\bar{p}}$ is determined from condition $V(\bar{\bar{p}} \mid p_1, \bar{\bar{p}}) = 0$. Therefore, benefits can only arise from pooling posteriors b_1 in the neighborhood of $\bar{p}$, and perfect communication is optimal for posteriors b_1 above the pooling region. The benefits of pooling come from inducing

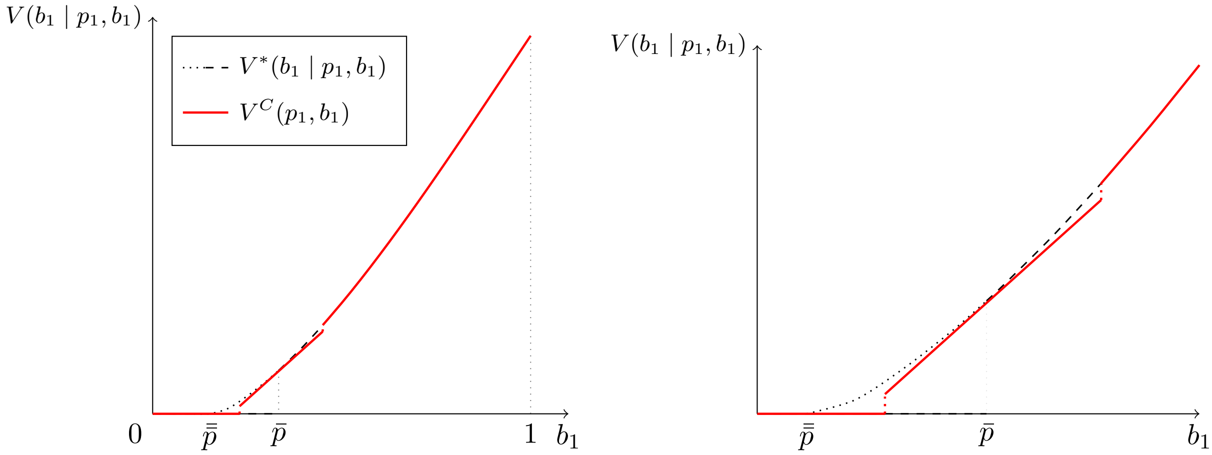

Figure 9: $V^*(b_1 \mid p_1, b_1)$ and the commitment value $V^C(p_1, b_1)$.
Note: $H = 3, L = 0, c = 1, \sigma = 2$. To illustrate convexity, it is also assumed for this graph that C1 cares about third consumer's utility 9 times as much as about C2's. The left panel plots functions for all $b_1 \in [0, 1]$; the right panel focuses on the neighboorhood of $\bar{p}$.

experimentation after $b_1 < \bar{p}$, which means that the posterior induced by pooling must exactly equal $\bar{p}$. Indeed, otherwise C1 can lower the cutoff above which perfect communication occurs, forcing C2 to buy the item for all the same $b_1 < \bar{p}$, and conveying better information for some $b_1 > \bar{p}$, which is an improvement.

Finally, it is always optimal to pool at least some private beliefs to the left and to the right of $\bar{p}$. Indeed, suppose that a consumer sends the same message m for all private beliefs ε-below and $C \cdot \varepsilon$-above $\bar{p}$ such that resulting $q(p_t, m) = \bar{p}$.[83] This is equivalent to substituting value from truth-telling in this interval with a line tangent to $V(b_1 \mid p_1, b_1)$ at $b_1 = \bar{p}$. The gains from it are approximately equal to $V(\bar{p} \mid p_1, \bar{p}) \cdot \varepsilon > 0$, because $V(\bar{p} \mid p_1, \bar{p}) > 0$. Losses associated with pooling above the cutoff are less then $\frac{C(H-L)}{2}\varepsilon^2 = \mathcal{O}(\varepsilon^2)$. Therefore, there always exists $\varepsilon > 0$ such that it is optimal to pool at least some small neighborhood of private beliefs b_t around $\bar{p}$. Figure 9 plots the value attained by C1 under this communication mechanism. Although this argument shows that it is always beneficial to pool private beliefs below and above $\bar{p}$, it does not however show *how* it is optimal to pool beliefs. It tuns out that in the general case the set of pooled private beliefs can take more complicated forms.

General Case

This section describes the equilibrium outomes and the commitment solution arising in the infinite-horizon version of the game. We show that neither of the two feature perfect communication, with the exception that it may be an equilibrium outcome in some period as long as future

[83] As distributions $\mu^\theta(b_t|p_t)$ are arbitrary the weights between two sides from $\bar{p}$ do not have to be equal.

communication is uninformative.

Commitment Solution

We begin by looking at the commitment solution this time around. The main result and the argument behind it mirror those that we have discovered in the three-period example: the reviewer's desire to inflate the review of a marginally-bad item for sake of social experimentation results in him pooling moderately bad experiences with some good experiences so as to push the posterior of the next consumer up to $\bar{p}$. After all other experiences the reviewer reports truthfully. This idea is formalized in Theorem 4 below.

Theorem 4. *At every public state p_t the optimal commitment solution is characterized by a cutoff $l^C(p_t) < \bar{p}$ and a set of points $I^C(p_t)$ above $\bar{p}$ such that:*

1. *for all $b_t \leqslant l(p_t)$ consumer sends message $m \in \mathcal{S}(p_t)$, i.e., experimentation stops.*
2. *for any $b_t \notin I^C(p_t)$ consumer truthfully transmits his private belief b_t (or what is the same his private payoff) to the public, that is $p_{t+1} = b_t$.*
3. *for all $b_t \in (l(p_t), p_t) \bigcup I^C(p_t)$ consumer sends message m such that $q(p_t, m) = \bar{p}$.*

The proof of the theorem in the appendix proceeds in three main steps. First, we show that $V(b_t \mid p_t, b_t)$ is continuously differentiable and [weakly] convex in b_t above $\bar{p}$. This implies that pooling is only beneficial around the cutoff. Since $V(p_t, \bar{p}) > 0$, there are potential gains from pooling posteriors $b_t < \bar{p}$ with posteriors above the cutoff. This, however, would at the same time decrease the quality of information transmitted after those $b_t > \bar{p}$ that are pooled with posteriors below the cutoff, which is socially costly. The second major step of the proof is thus in showing that the gains from pooling over an arbitrarily small interval of posteriors will be of first order, while the losses will be of second order. Finally, we show that the optimal commitment strategy exists within the class of strategies restricted to such combinations of pooling around the cutoff and truthtelling otherwise. This existence is proved with the help of Arzela-Ascoli Theorem. Furthermore, we show that the optimal strategy is Markovian, i.e., depends only on public belief p_t but does not explicitly depend on t.

It is straightforward that the commitment solution induces underexperimentation relative to the first best (in which the consumer can transmit the information perfectly while also having perfect control over the future consumers' actions). This is because experimentation is costly:

to provide incentives for future consumers to experiment with the product after $b_t < \bar{p}$, the sender must distort the information transmitted after $b_t \geqslant \bar{p}$. In particular, this distortion is downwards, meaning it makes all future consumers more pessimistic and so exacerbates the problem of underexperimentation after those histories. In other words, the reviewer in period t has to trade off underexperimentation at $t+1$ against underexperimentation from $t+2$ onwards.

It is also worth pointing out the differences between our model of commited sender and the model of Che and Hörner [2018]. One obvious difference lies in the signal structure, where we allow for a wide class of private signals compared to binary (good news Poisson) signals in their case. This allows us to give a richer characterization of within-period outcomes at the cost of the tractability of the dynamics in the model. However, a more substantial difference between the two models lies in the technologies: in the model of Che and Hörner [2018] the principal is long-lived, and has memory of old information even if it was not publicly disclosed at the time, so this information may be disclosed at a later time. In our model, in contrast, all consumers are short-lived, hence the public record of product reviews is the only past information available today. This constraint to a public memory limits the principal designing a reviewing mechanism to "now-or-never" revelation schemes, eliminating the opportunity to delay.

Decentralized Equilibrium

We now move on to exploring the equilibria of the decentralized game. The analysis is complicated by the fact that an equilibrium at t is determined by the continuation equilibria after various induced priors p_{t+1}, which in turn depend on continuation equilibria after p_{t+2} and so on. Backwards induction is not available in an infinite-horizon game, and even restricting ourselves to Markov setting, where strategies only depend on the public prior p but not calendar time t, does not render the problem tractable enough to provide a full characterization of the set of equilibria.

We are, however, able to provide a partial characterization of equilibria. In particular, Theorem 5 below provides two statements pertaining to such characterization. First, it claims that commitment is always valuable in the sense that no equilibrium of the decentralized game can generate a higher lever of social welfare. Second, it shows that experiences b_t in some neighborhood of the myopic cutoff $\bar{p}$ are always pooled together into a single review – this applies to any equilibrium of the game.

Theorem 5. *1. For any equilibrium in the decentralized game, $V^D(p_t) \leqslant V^C(p_t)$ for any p_t.*

2. For any p_t for which there exists a $p \in \mathcal{P}(p_t)$ that does not start a cascade, there exist $l^D(p_t)$,

$r^D(p_t) > l^D(p_t)$ and $m \in \mathcal{M}$ such that $\bar{p} \in [l(p_t), r(p_t)]$, and for all $b_t \in [l(p_t), r(p_t)]$ we have $r(m \mid p_t, b_t) = 1$.

The first statement is relatively straightforward, since the principal in the commitment scenario has access to any communication structure that can arise in the equilibrium of a decentralized game. The second statement mostly mirrors the intuition from the three-period example. The part that is worth pointing out is the qualifier on p_t: in particular, communication at p_t is noisy only if at least some message is available in $\mathcal{M}(p_t)$ that does not start a cascade. The complementary case was discussed in Proposition 18: if all messages in $\mathcal{M}(p_t)$ start a cascade then perfect comunication is possible.

Conclusion

This paper builds a theoretical model of social learning through product reviews, focusing on the issue of information provision. We look closely at the empirically observed tension between self-interest in purchasing behavior and prosocial motives when writing a review, and we investigate how this tension affects the informational content of the reviews. The conflict emerges from the reviewers' desire to deceive future consumers into buying a potentially subpar product for sake of generating information beneficial for the society.

We show that truthful communication through reviews cannot be sustained in the equilibrium of such a model. Moreover, despite the conflict only arising under specific circumstances, the noise created by it propagates, making *all* communication noisy in equilibrium. If, however, the reviewer can commit to a particular communication strategy before experiencing the product or, equivalently, a social norm can be chosen by a welfare-maximizing principal that all consumers will have to follow, then the noise in communication is more confined, and perfect communication is possible when the reviewer sees the product as very good.

This paper contributes to the broader literature on social learning, helping to identify the issues that can deteriorate the quality of learning via product reviews, demonstrating that even in the absence of any kind of interference from the firm, reviews may not be the perfect source of information about products with uncertain characteristics.

Rigid Experimentation[84]

Introduction and Relation to the Literature

Implementing risky alternative of *ex ante* unknown quality is sometimes hard to reverse. Consider a firm who decides on whether to launch its chain of stores in a new country. It does not initially know whether their product will match the local tastes. However, the only way to know the answer is to actually enter the new market. Expanding and maintaining capacity is costly and can eventually be not worth it in case the product does not match the local tastes. Thus the firm should potentially be cautious in expanding its capacity. But at the same time larger capacity means faster learning about the quality of the match and therefore provides incentives to invest into it. Within this paper we examine how optimal expansion path looks like for a single firm and when multiple firm strategically interact and have perfectly (positively) correlated demands for their products.

We contribute to three strands of literature. First strand, represented by now-classical papers by Bolton and Harris [1999] and Keller, Rady, and Cripps [2005], deals with experimentation models with pure information extenality. In these models in each moment an agent chooses an intensity of experimentation. The higher this intensity is, the more information the agent receives. However actions taken by the agent in previous periods do not restrict his choice today. In contrast, this paper builds a model where intensity choices can only be slightly adjusted through time. This creates a correlation between intensities of experimentation in any two moments in time. The rigidity in the intensity is modeled as an upper bound on the speed with which the intensity of experimentation can be increased. For simplicity we also assume that the agent can not decrease its intensity even when it becomes pessimistic enough about the quality of the risky option. The closest to this paper is the one by Julia Salmi and Murto [2019]. They study the same question, but model the news process with a Brownian motion rather than a Poisson process.

Second, the model features similarities with to the problem of when to exercise an option. In irreversible investment setting this question is extensively studied by Dixit and Pindyck [1994]. Daley and Green [2012], similarly, examine when a seller should sell a good of privately-known quality to a market of uninformed buyers when price for it evolves stochastically. The difference with these papers is that an action today not only affect the payoff the agent receives today, but also affects the speed with which an underlying uncertainty is resolved.

[84] This paper should be cited as A. Smirnov. Rigid Experimentation. mimeo, 2020.

Finally, the paper contributes to the literature on infant industries development, see for instance Caplin and Leahy [1993], Rob [2001], Rob and Vettas [2003], Decamps and Mariotti [2004].

The Model

Within the current section we present a model of a single firm. Firm starts to operate with zero initial capacity on the market with unknown demand. The demand can be of two types: *good* or *bad*. If demand is bad there are no consumers for the firm's product, whilst if it is good and the installed capacity of the firm is $\alpha \geqslant 0$ consumers willing to buy the good arrive as a Poisson process with intensity $\alpha\lambda$ where $\lambda \geqslant 0$ is a parameter of the model. From each purchase the firm receives a fixed benefit normalized to 1. Initial belief held by the firm that the demand is good is p_0. The firm can irreversibly invest into its capacity. Per unit of installed capacity the firm incurs flow maintenance costs of amount c. To capture the riskiness of investments and to rule out trivial cases we assume that $\lambda > c > 0$. Time is continuous and the discount rate is r.

At every point in time the firm

1. observes the current state variable - a pair (α, p) where α is already installed capacity and p is the current belief that the demand is good.

2. decides whether to expand the capacity and by how much up to the limit of $\bar{i}$. We allow only for marginal increases in capacity, in other words the firm cannot increase its capacity by more that $\bar{i} \times dt$ in a time interval of the length dt. For simplicity of exposition we aslo assume that the firm cannot decrease its capacity.

3. observes whether a consumer has arrived and updates its belief about the demand accordingly.

In the baseline model we allow the firm to decide only on whether to invest into capacity not allowing the firm to exit the market when its belief about the demand becomes pessimistic enough.

Equilibrium Analysis

Belief Update

If in the beginning of the period of the length dt firm's state variable is (α, p) and it has invested a total amount of Idt into its capacity in it, a consumer arrives with subjective probability of $p(1 - e^{-(\alpha+Idt)\lambda dt}) \approx p\alpha\lambda dt$. As that event perfectly reveals the demand quality the belief instantly jumps to $p = 1$. If however no consumer has arrived, which happens with probability

$(1-p)+pe^{-(\alpha+Idt)\lambda dt} \approx 1-p\alpha\lambda dt$, firm's updated belief at the end of the period is

$$p+dp = \frac{(1-\alpha\lambda dt)p}{(1-\alpha\lambda dt)p+(1-p)}$$

according to Bayes' rule. Simplifying and canceling higher-order terms we arrive to the well-known expression

$$dp = -\lambda\alpha p(1-p)dt.$$

Note however the difference in interpretation of α in our model and the model of Keller, Rady, and Cripps [2005]. Within latter it is an *instant* intensity of experimentation whilst in our model it is an *accumulated* intensity.

Maximization Problem

Denote by $V(\alpha,p)$ associated with the state value function. The objective of the firm is to choose investment plan $\{I_t\}_{t\geq 0}$ that maximizes

$$\mathbb{E}\left[\int_0^{+\infty} re^{-rt}\alpha_t\left(\lambda p-c\right)dt\,\middle|\,\alpha,p\right]$$

subject to the investment constraints

$$\dot{\alpha_t} = I_t,$$
$$0 \leqslant I_t \leqslant \bar{i},$$

and the evolution of belief. Following the literature on strategic experimentation we express the value in per-period terms multiplying all flow payoffs by r.

When $p=0$ or $p=1$ the firm knows the quality of the demand with certainty and therefore its decision is based solely on the quality of demand, but not on the learning externalities generated by the capacity. With $p=0$ and arbitrary α the firm disinvests and therefore

$$V(\alpha,0) = -\int_0^{+\infty} re^{-rt}\alpha c dt = -\alpha c. \tag{25}$$

With $p = 1$ the firm always invests and accumulates ever growing capacity:

$$V(\alpha, 1) = \int_0^{+\infty} re^{-rt}(\alpha + \bar{i}t)(\lambda - c)\, dt = \left(\alpha + \frac{\bar{i}}{r}\right)(\lambda - c). \tag{26}$$

By the Principle of Optimality, the value function satisfies the "discrete version" of Hamilton-Jacobi-Bellman equation

$$V(\alpha, p) = \max_I \left[-r\alpha c dt + p\alpha\lambda dt(r + e^{-rdt}V(\alpha, 1)) + (1 - p\alpha\lambda dt)(e^{-rdt}(V(\alpha + Idt, p + dp))\right].$$

Within the period of time of the length dt with subjective probability $p(\alpha + Idt)\lambda dt \approx p\alpha\lambda dt$ a consumer arrives and the firm earns lump-sum profit of $r \cdot 1$ from the transaction, pays its flow costs of $r \cdot (\alpha + Idt)cdt \approx r\alpha cdt$ and shifts to the new state from the "next" moment. If however no consumer has arrived the firm pays flow costs and updates its belief about the market accordingly.

Substituting $V(\alpha, 1)$ from (26), using $1 - rdt$ to approximate e^{-rdt} and canceling out the terms of order dt^2 and higher we obtain the HJB equation for the value function:

$$(r + \alpha\lambda p)V(\alpha, p) = \max_{I_t} \left[\alpha\lambda p\left(r + \left(\alpha + \frac{\bar{i}}{r}\right)(\lambda - c)\right) - \alpha cr + I_t V'_\alpha(\alpha, p) - \alpha\lambda p(1 - p)V'_p(\alpha, p)\right].$$

The resulting investment rule is particularly simple: invest $\bar{i}$ if and only if $V'_\alpha(\alpha, p) > 0$, do not invest when $V'_\alpha(\alpha, p) < 0$ and invest any $I_t \in [0, \bar{i}]$ when $V'_\alpha(\alpha, p) = 0$.

Disinvestment Region

If $I_t^* = 0$ we do not have term $V'_\alpha(\alpha, p)$ and therefore we can treat α as a parameter in a first-order linear differential equation

$$(r + \alpha\lambda p)V = \left[\alpha\lambda p\left(r + \left(\alpha + \frac{\bar{i}}{r}\right)(\lambda - c)\right) - \alpha cr - \alpha\lambda p(1 - p)V'_p\right]. \tag{27}$$

General solution to it is given by

$$V_0^S(\alpha, p) = \alpha(\lambda p - c) + \frac{\alpha\lambda(\lambda - c)\bar{i}}{r(\alpha\lambda + r)}p + K(\alpha)(1 - p)\left(\frac{1 - p}{p}\right)^{\frac{r}{\alpha\lambda}},$$

where $K(\alpha)$ is an arbitrary differentiable function. From (25) we infer that $K(\alpha) \equiv 0$, i.e. the value function is linear in p in the no-investment region.

Investment Region

When $I_t^* = \bar{i}$ we obtain a first-order linear partial differential equation

$$(r + \alpha\lambda p)V = \left[\alpha\lambda p\left(r + \left(\alpha + \frac{\bar{i}}{r}\right)(\lambda - c)\right) - \alpha c r + \bar{i}V'_\alpha - \alpha\lambda p(1-p)V'_p\right]. \tag{28}$$

To solve it we need to find its particular solution and a first integral of a modified homogeneous equation.[85] The final answer is given by

$$V_1^S(\alpha, p) = \left(\alpha + \frac{\bar{i}}{r}\right)(\lambda p - c) + pe^{\frac{1}{\bar{i}}\left(r\alpha + \lambda\frac{\alpha^2}{2}\right)} G_1\left(\frac{1}{\lambda}\ln\left(\frac{1-p}{p}\right) - \frac{\alpha^2}{2\bar{i}}\right),$$

where $G_1(\cdot)$ is an arbitrary differentiable function.

Equilibrium Conditions

In this section we summarize the optimal behavior of a single firm. Our approach relies on a standard verification argument resulting in a simple threshold strategy. In our framework threshold strategy is a function $p^S(\alpha)$ such that in state (α, p) it is optimal to invest if and only if $p > p^S(\alpha)$.

Strictly speaking, verification argument is not fully applicable in our framework as resulting value functions are unbounded in α. It happens because upon a breakthrough the firm is certain about the demand quality and its capacity becomes arbitrarily large resulting in linearly growing value function. However for any $p_0 < 1$ there is a maximal capacity a firm can accumulate *on the equilibrium path* not yet achieving a breakthrough. That implies that before a breakthrough found V replicates the optimal decision making whilst after the breakthrough the optimal decision making is evident without any reference to optimal control. More formally one can consider the same Bellman equation with modified $\tilde{V} = V - \alpha(\lambda p - c)$, solution to which will always be bounded.

[85] To be more precise, first we eliminate non-homogeneous summand and then make a substitution $V(\alpha, p) = pe^{r\alpha + \lambda\frac{\alpha^2}{2}}\tilde{V}(\alpha, p)$ which eliminates the LHS. Finally we guess the first integral of the resulting equation.

To find the equilibrium we induce the following set of conditions:

$$\begin{aligned}
V_0^S(\alpha, p^S(\alpha)) = V_1^S(\alpha, p^S(\alpha)) &\quad (ValueMatching)\\
V_{0p}'^S(\alpha, p^S(\alpha)) = V_{1p}'^S(\alpha, p^S(\alpha)) &\quad (Smooth-Pasting)\\
V_{0\alpha}'^S(\alpha, p) \leqslant 0 \text{ for all } p \leqslant p^S(\alpha) &\quad (LeftOptimality)\\
V_{1\alpha}'^S(\alpha, p) \geqslant 0 \text{ for all } p \geqslant p^S(\alpha) &\quad (RightOptimality)\\
V_{0\alpha}'^S(\alpha, p^S(\alpha)) = 0 \qquad &\quad (InvestmentSmooth-PastingLeft)\\
V_{1\alpha}'^S(\alpha, p^S(\alpha)) = 0 \qquad &\quad (InvestmentSmooth-PastingRight)
\end{aligned}$$

First four conditions are standard for the experimentation literature. The last two emerge because of two-dimensional structure of the state variable. Although they are not needed for optimality of an investment strategy, we show that they both hold. Additionally, $(IS-PL)$ significantly easens the calculation of the threshold function $p^S(\alpha)$.

Lemma 3. *$(IS-PR)$ follows from (VM) and (SP).*

Proof. Equations (27) and (28) differ only in one summand. If both (VM) and (SP) are satisfied it automatically implies the remaining summand must be zero. □

For now we ignore (LO) and (RO) conditions. We will check them explicitly afterwards. Therefore in total we have three equations ((VM), (SP) and $(IS-PL)$) for two unknowns: the threshold function $p^S(\alpha)$ and function $G(\cdot)$. We first look at $(IS-PL)$ and calculate the resulting threshold. Then we solve (VM) and (SP) system separately and verify that the resulting threshold actually coincides with the one obtained from $(IS-PL)$.

$(IS-PL)$ implies that

$$-c + \lambda p + \frac{\lambda(\lambda - c)\bar{i}}{r} \frac{r}{(\alpha\lambda + r)^2} p = 0 \tag{29}$$

or

$$p^S(\alpha) = \frac{c}{\lambda + \frac{\lambda(\lambda-c)\bar{i}}{(\alpha\lambda+r)^2}} \tag{30}$$

Finally note that LHS in (29) is increasing in p and therefore (LO) is satisfied. Now turn to (VM)

and (SP). They constitute the following system of equations:[86]

$$\alpha(\lambda p - c) + \frac{\alpha\lambda(\lambda - c)\bar{i}}{r(\alpha\lambda + r)}p = \left(\alpha + \frac{\bar{i}}{r}\right)(\lambda p - c) + (1-p)e^{\frac{r}{\bar{i}}\alpha}G,$$
$$\alpha\lambda + \frac{\alpha\lambda(\lambda - c)\bar{i}}{r(\alpha\lambda + r)} = \left(\alpha + \frac{\bar{i}}{r}\right)\lambda - e^{\frac{r}{\bar{i}}\alpha}G - (1-p)e^{\frac{r}{\bar{i}}\alpha}\frac{1}{\lambda p(1-p)}G'.$$

We first solve for G and G'.

$$\frac{c\bar{i}}{r} - \frac{(\lambda - c)\,\bar{i}}{\alpha\lambda + r}\frac{p}{1-p} = e^{\frac{r}{\bar{i}}\alpha}G \tag{31}$$

$$\frac{(\lambda - c)\,\bar{i}}{\alpha\lambda + r}\frac{p}{1-p} = \frac{1}{\lambda}e^{\frac{r}{\bar{i}}\alpha}G' \tag{32}$$

We can treat (31) as an equality where functions of α are on both sides and, as they coincide as functions, to equate their derivatives (we can do the same with (SP), but there is no good reason for that). Also for simplicity let us denote $s := \frac{p}{1-p}$ - the odds ratio. Then we get

$$-\frac{(\lambda - c)\,\bar{i}}{\alpha\lambda + r}s' + \frac{\lambda\,(\lambda - c)\,\bar{i}}{(\alpha\lambda + r)^2}s = \frac{r}{\bar{i}}e^{\frac{r}{\bar{i}}\alpha}G - \frac{\alpha}{\bar{i}}e^{\frac{r}{\bar{i}}\alpha}G' + e^{\frac{r}{\bar{i}}\alpha}G'\left(-\frac{s'}{\lambda s}\right). \tag{33}$$

We can substitute the values of G and G' from (31) and (32) into (33) to obtain the exact formula for $s(\alpha)$:

$$-\frac{(\lambda - c)\,\bar{i}}{\alpha\lambda + r}s' + \frac{\lambda\,(\lambda - c)\,\bar{i}}{(\alpha\lambda + r)^2}s = \frac{r}{\bar{i}}\left(\frac{c\bar{i}}{r} - \frac{(\lambda - c)\,\bar{i}}{\alpha\lambda + r}s\right) - \left(\frac{\alpha}{\bar{i}} + \frac{s'}{\lambda s}\right)\frac{\lambda\,(\lambda - c)\,\bar{i}}{\alpha\lambda + r}s.$$

As can be easily seen, s' cancels out and we obtain an expression for $s(\alpha)$ from which we easily restore $p^S(\alpha)$. Occasionally, the resulting threshold coincides with the one obtained earlier in (30). The following proposition summarizes the established fact.

Proposition 19. *Let the state variable for the single-firm problem be (α, p). Then the firm invests $\bar{i}$ if $p > p^S(\alpha) = \frac{c}{\lambda + \frac{\lambda(\lambda - c)\bar{i}}{(\alpha\lambda + r)^2}}$ and does not invest if $p \leqslant p^S$.*

Figure 10 illustrates the optimal strategy. The firm starts with zero initial capacity on the horizontal axis and gradually accumulates it. If a breakthrough is achieved then the firm eternally accumulates capacity. If, conversely, no breakthrough is achieved until the orange threshold is reached in state space (p, α) then the firm stops further investments.

[86] We additionally rewrite the option value redefining $G(x) = e^{-\lambda x}G_1(x)$

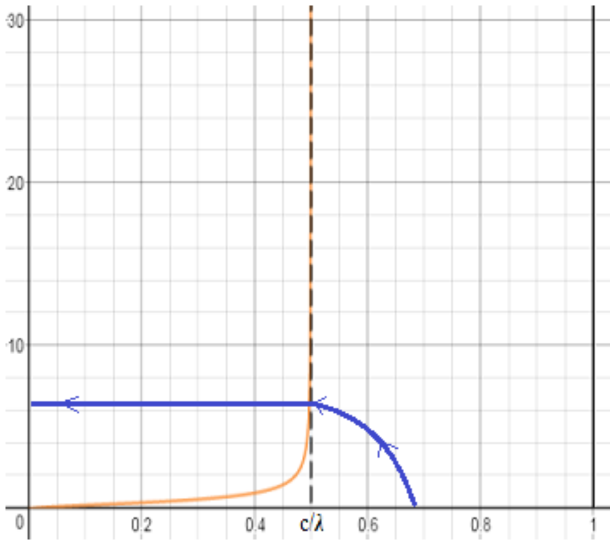

Figure 10

Oligopoly Problem

In the next two sections we enrich our model with a possibility of interaction between several firms. First, as a benchmark model, we solve the *cooperative problem* of N firms all operated by a welfare-maximizing social planner. Then we compare it with the *strategic problem* where each of the firms makes an individualistic decision to identify potential inefficiencies of a decentralized equilibrium.

For both scenarios we first clarify the timing and informational structure of the game particularly stressing the differences with a single-firm framework.

At every point in time each of N firms

1. observes the current state variable - a pair $(\overline{\alpha}, p)$ where $\overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is an N-*dimensional vector of already installed capacities of all firms* and p is the current belief that the demand is good.

2. decides whether to expand the capacity and by how much up to the limit of $\bar{i}$. The firm cannot decrease its capacity.

3. observes whether a consumer has arrived *to any firm* and updates its belief about the demand quality accordingly.

Similar to the discrete version of the Bellman equation for a single firm we can write one for the individual firm in the presence of $N-1$ other firms. Define $\alpha := \sum_{i=1}^{N} \alpha_i$, $\alpha_{-i} := \alpha - \alpha_i$ and

$\overline{\alpha}_{-i} := (\alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \alpha_N)$. Then

$$V_i(\alpha_i, \overline{\alpha}_{-i}, p) = \max_{I_i} \left[p\alpha_i\lambda r dt - r\alpha_i c dt + p\alpha\lambda dt(e^{-rdt}V(\alpha_i, 1)) + (1 - p\alpha\lambda dt)e^{-rdt}V(\alpha_i + I_i dt, \overline{\alpha}_{-i} + \overline{I}_{-i}dt, p + dp) \right].$$

Canceling out the terms of order dt^2 and higher, substituting the belief update and $V(\alpha_i, 1)$ from (26) we obtain the HJB equation for the value function in oligopoly problem:

$$(r + \alpha\lambda p)V_i = \max_{I_i} \left[\alpha\lambda p \left(\alpha_i + \frac{\bar{i}}{r} \right) (\lambda - c) + r\alpha_i (\lambda p - c) + I_i V'_{i,\alpha_i} + \sum_{j \neq i} I_j V'_{i,\alpha_j} - \alpha\lambda p(1-p)V'_{i,p} \right] \tag{34}$$

Cooperative Solution

Within this section we assume that all firms are operated by the social planner who maximizes the total welfare (the sum of individual value functions). Therefore summing up (34) among individual firms we obtain

$$(r + \alpha\lambda p)V = \max_{I_1, I_2, \ldots, I_N} \left[\alpha\lambda p \left(\alpha + \frac{N\bar{i}}{r} \right) (\lambda - c) + r\alpha (\lambda p - c) + \left(\sum_{i=1}^{N} I_i \right) V'_{\alpha_i} - \alpha\lambda p(1-p)V'_p \right],$$

where $V := \sum\limits_{i=1}^{N} V_i$. Due to the symmetry of the firms to the social planner V'_{α_i} does not depend on i and therefore V depends only on aggregate installed capacity. Consequently, we can treat it as a 2-dimensional function of (α, p) instead of an $(N+1)$-dimensional function, reducing the problem to the one considered in the previous section. Going through the same steps as in the single-firm problem we establish the following proposition.

Proposition 20. *Let the state variable for the cooperative oligopoly problem be $(\overline{\alpha}, p)$ and $\alpha = \sum\limits_{i=1}^{N} \alpha_i$. Then all firms invest $\bar{i}$ if $p > p^{OC}(\alpha) = \frac{c}{\lambda + \frac{N\lambda(\lambda - c)\bar{i}}{(\alpha\lambda + r)^2}}$ and do not invest if $p \leqslant p^{OC}$.*

Strategic Solution

Within this section we identify unique symmetric equilibrium of the strategic problem. Similar to the single-firm problem we can easily write a best-response correspondence for a firm.

$$I_i^* \begin{cases} = \bar{i} & \text{if } V'_{i,\alpha_i} > 0 \\ \in [0, \bar{i}] & \text{if } V'_{i,\alpha_i} = 0 \\ = 0 & \text{if } V'_{i,\alpha_i} < 0 \end{cases}$$

As we are looking for a symmetric equilibrium we assume that all firms but firm i exploit the same strategy and deduce the value functions assuming the firm itself then follows the same strategy. Finally, we verify the optimality of firm i's strategy.

Case 1: $I_i = 0$, $I_j = 0$ for all $j \neq i$.

Then the general solution to (34) is given by

$$V_0^{OS}(\alpha_i, \overline{\alpha}_{-i}, p) = \alpha_i(\lambda p - c) + \frac{\alpha\lambda(\lambda - c)\overline{i}}{r(\alpha\lambda + r)} p + K(\alpha_i, \overline{\alpha}_{-i})(1-p)\left(\frac{1-p}{p}\right)^{\frac{r}{\lambda\alpha}}.$$

As before due to the boundary condition $K_2(\alpha_i, \overline{\alpha}_{-i}) \equiv 0$, so we have a linear in p value function.

Case 2: $I_i = \overline{i}$, $I_j = \overline{i}$ for all $j \neq i$.

The general solution to (34) is then given by (note that now G is a two-dimensional function)

$$V_1^{OS}(\alpha_i, \overline{\alpha}_{-i}, p) = \left(\alpha_i + \frac{\overline{i}}{r}\right)(\lambda p - c) + (1-p)e^{\frac{r}{\overline{i}}\alpha_i} G\left(\alpha_i - \frac{\alpha_{-i}}{N-1}, \frac{1}{\lambda}\ln\left(\frac{1-p}{p}\right) - \frac{\alpha_i^2}{2\overline{i}} - \sum_{j\neq i}\frac{\alpha_j^2}{2\overline{i}}\right).$$

Case 3: $I_i = I_j = I \in (0, \overline{i})$ for all $j \neq i$.

In the *mixing region* any player is indifferent between investing or not and therefore $V'_{i,\alpha_i}(\alpha_i, \overline{\alpha}_{-i}, p) = 0$ implying the value function in it does not depend on α_i. Denote the general solution for this region as $M^{OS}(\overline{\alpha}_{-i}, p)$.

Equilibrium construction

In equilibrium the following set of conditions hold:[87]

$$\begin{aligned}
V_0^{OS}(\alpha_i, \overline{\alpha}_{-i}, p^L(\overline{\alpha})) &= M^{OS}(\alpha_i, \overline{\alpha}_{-i}, p^L(\overline{\alpha})) && (ValueMatchingLeft)\\
V_{0p}'^{OS}(\alpha_i, \overline{\alpha}_{-i}, p^L(\overline{\alpha})) &= M_p'^{OS}(\alpha_i, \overline{\alpha}_{-i}, p^L(\overline{\alpha})) && (Smooth-PastingLeft)\\
V_{0\alpha_i}'^{OS}(\alpha_i, \overline{\alpha}_{-i}, p) \leqslant 0 &\text{ for all } p \leqslant p^L(\overline{\alpha}) && (LeftOptimality)\\
M^{OS}(\alpha_i, \overline{\alpha}_{-i}, p^R(\overline{\alpha})) &= V_1^{OS}(\alpha_i, \overline{\alpha}_{-i}, p^R(\overline{\alpha})) && (ValueMatchingRight)\\
M_p'^{OS}(\alpha_i, \overline{\alpha}_{-i}, p^R(\overline{\alpha})) &= V_{1p}'^{OS}(\alpha_i, \overline{\alpha}_{-i}, p^R(\overline{\alpha})) && (Smooth-PastingRight)\\
V_{1\alpha_i}'^{OS}(\alpha_i, \overline{\alpha}_{-i}, p) \geqslant 0 &\text{ for all } p \geqslant p^R(\overline{\alpha}) && (RightOptimality)
\end{aligned}$$

[87] Note that we demand these equalities to hold only *on the equilibrium path* where $\alpha_i = \alpha/N$ for all i. Otherwise we cannot exploit the structure of symmetric equilibrium.

As in the single-firm problem we impose additional smooth-pasting conditions on both thresholds:

$$V'^{OS}_{0\alpha_i}(\alpha, p^L(\overline{\alpha})) = 0 \quad (InvestmentLeftSmooth - Pasting)$$
$$V'^{OS}_{1\alpha_i}(\alpha, p^R(\overline{\alpha})) = 0 \quad (InvestmentRightSmooth - Pasting)$$

What is even more surprising, although we do not demand it, resulting value function will eventually be continuously-differentiable with respect to all $\overline{\alpha}_{-i}$ as well.

Left threshold

As in single-firm problem we can obtain the first threshold from $(IS - PL)$. Differentiating V_0^{OS} with respect to α_i we get

$$-c + \lambda p + \frac{\lambda(\lambda - c)\bar{i}}{r} \frac{r}{(\alpha\lambda + r)^2} p = 0$$

or

$$p^L(\overline{\alpha}) = \frac{c}{\lambda + \frac{\lambda(\lambda - c)\bar{i}}{(\alpha\lambda + r)^2}}. \tag{35}$$

Note that the decision on whether to invest some positive amount depends for any firm only on aggregate installed capacity, but not on individual capacities.

Alternatively, the system of (VM) and (SP) implies

$$\alpha_i(\lambda p - c) + \frac{\alpha\lambda(\lambda - c)\bar{i}}{r\,(\alpha\lambda + r)} p = M(\alpha_{-i}, p),$$
$$\alpha_i\lambda + \frac{\alpha\lambda(\lambda - c)\bar{i}}{r\,(\alpha\lambda + r)} = M'_p(\alpha_{-i}, p).$$

We can take the full derivative of the first equality with respect to α_i and get:

$$\lambda p - c + \alpha_i\lambda \frac{\partial p}{\partial \alpha_i} + \frac{\lambda(\lambda - c)\bar{i}}{(\alpha\lambda + r)^2} p + \frac{\alpha\lambda(\lambda - c)\bar{i}}{r\,(\alpha\lambda + r)} \frac{\partial p}{\partial \alpha_i} = M'_p \frac{\partial p}{\partial \alpha_i}.$$

All terms with $\frac{\partial p}{\partial \alpha_i}$ cancel out and we obtain the expression for $p^L(\overline{\alpha})$ which coincides with (35).

Intermediate Value

To find the right threshold we first need to find resulting value function in the interior region. It can be done using (VM_L) condition.

$$\alpha_i(\lambda p^L(\overline{\alpha}) - c) + \frac{\alpha\lambda(\lambda - c)\bar{i}}{r\,(\alpha\lambda + r)} p^L(\overline{\alpha}) = M(\alpha_{-i}, p^L(\overline{\alpha})).$$

From (35) we get

$$\lambda p^L(\overline{\alpha}) - c + \frac{\lambda(\lambda - c)\bar{i}}{(\alpha\lambda + r)^2} p^L(\overline{\alpha}) = 0,$$

and therefore

$$M(\alpha_{-i}, p^L(\overline{\alpha})) = \alpha_i(\lambda p^L(\overline{\alpha}) - c) - \frac{\alpha\,(\alpha\lambda + r)}{r}(\lambda p^L(\overline{\alpha}) - c) = -(\lambda p^L(\overline{\alpha}) - c)\left(\alpha_{-i} + \frac{\lambda}{r}\alpha^2\right).$$

Substituting α from (35) into the expression above we finally obtain

$$M(\alpha_{-i}, p) = -(\lambda p - c)\left(\alpha_{-i} + \frac{r}{\lambda}\right) + \frac{\lambda - c}{r} p - 2\sqrt{\frac{\lambda - c}{\lambda} p(c - \lambda p)}.$$

Note that the function is convex in p (as it should be in models with positive value of information).

Right threshold

To identify the second threshold we need to solve the following system of equations:

$$\begin{aligned}
\left(\alpha_i + \frac{1}{r}\right)(\lambda p_2 - c) + (1 - p_2)e^{r\alpha_i} G &= M, \\
(\lambda p_2 - c) + r(1 - p_2)e^{r\alpha_i} G + (1 - p_2)e^{r\alpha_i} G_1' - \alpha_i(1 - p_2)e^{r\alpha_i} G_2' &= 0, \\
\left(\alpha_i + \frac{1}{r}\right)\lambda - e^{r\alpha_i} G - (1 - p_2)e^{r\alpha_i} G_2' \frac{1}{\lambda p_2(1 - p_2)} &= M_2', \\
-(1 - p_2)e^{r\alpha_i} G_1' - \alpha_{-i}(1 - p_2)e^{r\alpha_i} G_2' &= M_1'.
\end{aligned}$$

Solving the system we get the familiar equality which determines the second cut-off.

$$(r + (\alpha_i + \alpha_{-i})\,\lambda p)M = \left[(\alpha_i + \alpha_{-i})\,\lambda p\left(\alpha_i + \frac{1}{r}\right)(\lambda - c) + r\alpha_i\,(\lambda p - c) + M_1' - (\alpha_i + \alpha_{-i})\,\lambda p(1 - p)M_2'\right].$$

We substitute the linear and the quadratic parts of M separately.

$$(\lambda - c)p - \frac{r^2}{\lambda}(\lambda p - c) + (\lambda p - c) - r\alpha(\lambda p - c) - \alpha p(\alpha\lambda + r)(\lambda - c) - 2r\sqrt{\frac{\lambda - c}{\lambda} p(c - \lambda p)} - \alpha\sqrt{\frac{p\lambda(\lambda - c)}{c - \lambda p}}\,(c + cp - 2\lambda p) = 0$$

We need to show that the second threshold exists and is in between p_1 and $\frac{c}{\lambda}$ (myopic experimentation cutoff). For that we first evaluate the expression in both endpoints. Second we verify that the above function is convex in p.

Case 1. $\alpha = 0$ (i.e. we are at the beginning of the game). In that case almost all terms cancel out.

$$(\lambda - c)p - \frac{r^2}{\lambda}(\lambda p - c) + (\lambda p - c) - 2r\sqrt{\frac{\lambda - c}{\lambda} p(c - \lambda p)} = 0.$$

At $\frac{c}{\lambda}$ the expression is positive whilst at p_1 we need additional calculations. In case of zero installed investments

$$p_1 = \frac{c}{\lambda + \frac{\lambda(\lambda - c)}{r^2}}.$$

Substituting we obtain

$$(\lambda - c)p - \frac{r^2}{\lambda}(\lambda p - c) + (\lambda p - c) - 2(\lambda - c)p = -(\lambda - c)p + \left(\frac{r^2}{\lambda} - 1\right)\frac{\lambda(\lambda - c)}{r^2}p < 0.$$

To verify the function is convex we need to calculate the second derivative of the square root:

$$\left(\sqrt{p(c - \lambda p)}\right)'' = \left(\frac{c - 2\lambda p}{2\sqrt{p(c - \lambda p)}}\right)' = \frac{\sqrt{p(c - \lambda p)}(-2\lambda) - (c - 2\lambda)^2 \frac{1}{2\sqrt{p(c - \lambda p)}}}{2p(c - \lambda p)} < 0.$$

Case 2. $\alpha > 0$. In that case we have a non-zero summand where we divide by zero. When $p \to \frac{c}{\lambda}$ from below the whole expression converges to $+\infty$ (it was not so in Case 1). Indeed, $c + cp_m - 2\lambda p_m = c + \frac{c^2}{\lambda} - 2c < 0$. Now we evaluate the expression at p_1. The first summand remains from the Case 1 so we do not write it.

$$r\alpha(c - \lambda p) - \alpha p(\lambda - c)(\alpha\lambda + r) - \alpha(\alpha\lambda + r)(c - \lambda p) + \alpha p(\alpha\lambda + r)(\lambda - c) = -\alpha^2\lambda(c - \lambda p) < 0.$$

Finally we check the remaining non-linear part of the equation is a convex function.

$$\begin{aligned}\left(\sqrt{\frac{p}{c - \lambda p}}(c + cp - 2\lambda p)\right)'' &= \frac{c}{\sqrt{p(c - \lambda p)^3}}(c - 2\lambda) - \frac{1}{4}(c + cp - 2\lambda p)\frac{c - 4\lambda p}{p(c - \lambda p)}\frac{c}{\sqrt{p(c - \lambda p)^3}} = \\ &= \frac{c}{4\sqrt{p(c - \lambda p)^3}}\frac{3cp - 2\lambda p - c}{p(c - \lambda p)} < 0.\end{aligned}$$

Lemma 4. *Let $f \in C^2[a, b]$ be a strictly convex function and $f(a) < 0 < f(b)$. Then there is a unique solution to the equation $f(x) = 0$ on $[a, b]$.*

Finally we can summarize the structure of the unique symmetric equilibrium in strategic

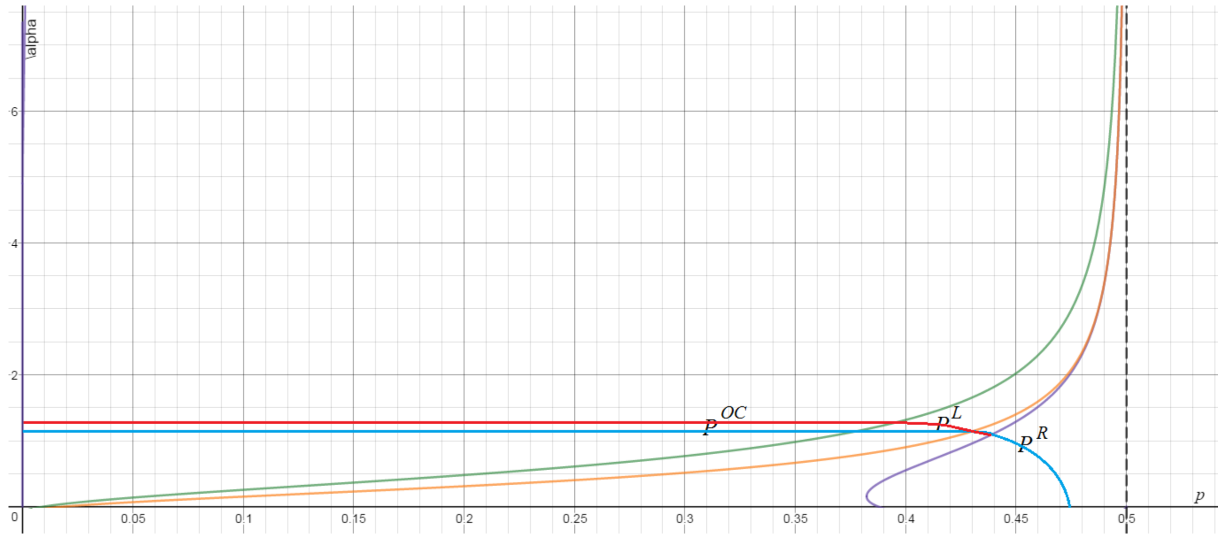

Figure 11

problem.

Proposition 21. *Let the state variable for the strategic oligopoly problem be* $(\overline{\alpha}, p)$ *and* $\alpha = \sum_{i=1}^{N} \alpha_i$. *Then there exist two functions* $p^L(\alpha)$ *and* $p^R(\alpha)$ *such that all firms invest* $\bar{i}$ *if* $p \geqslant p^R(\alpha)$, *do not invest if* $p \leqslant p^L(\alpha)$ *and invest some positive amount* $I(p) \in (0, \bar{i})$ *if* $p^L(\alpha) \leqslant p \leqslant p^R(\alpha)$. $p^L(\alpha) = \frac{c}{\lambda + \frac{\lambda(\lambda - c)\bar{i}}{(\alpha\lambda + r)^2}}$ *and* $p^R(\alpha)$ *solves*

$$\left(1 - \alpha^2\lambda - \alpha r\right)(\lambda - c)p - \left(\frac{r^2}{\lambda} - 1 + r\alpha\right)(\lambda p - c) - 2r\sqrt{\frac{\lambda - c}{\lambda} p(c - \lambda p)} - \alpha\sqrt{\frac{p\lambda(\lambda - c)}{c - \lambda p}}(c + cp - 2\lambda p) = 0$$

for every $\alpha \geqslant 0$.

Figure 11 illustrates both efficient cooperative solution (red line) and strategic solution (blue line). In both scenarios all firms starts with zero initial capacity. Until its expansion curve reaches threshold p^R in both cases the firm invests into its capacity at a maximal speed of $\bar{i}$. Then in between thresholds p^R and p^L in cooperative problem the firm still continues to invest at a maximal speed whilst in strategic framework gradually invests less and less. This happens because information provided by the other firms is a public good and a standard free-riding problem occurs. Finally, below p^L in both scenarios firms stop investments.

Conclusion

This paper presents a model of strategic experimentation with rigidity in experimentation's intensity. Existing literature mostly concentrates on frameworks where the choice of today's actions generate information only today. In contrast we make actions taken today to be irreversible and therefore to affect the experimentation in all consequent periods.

We solve for optimal experimentation strategy for a single agent and two problems with multiple agents: cooperative and strategic. We find that in decentralized equilibrium (strategic problem) the agents start to free-ride on information provided by other agents from some point onward which leads to underinvestment in capacity.

Part III

Appendices

Appendix for "Timing of Predictions in Dynamic Cheap Talk: Experts vs. Quacks"

Proof of Proposition 1. The proof is valid for all $\pi \in \left(\frac{1}{2}, 1\right]$. We show that $r^E(m,t) > 0$ if and only if $r^Q(m,t) > 0$ for any (m,t) for any history with $b(h_t^p) \in (0,1)$. Together with the fact that $b_0 \in (0,1)$, this will then mean that on equilibrium path we never arrive at a [non-terminal] history with $b(h_t^p) \in \{0,1\}$, hence the statement is true for all histories on equilibrium path.

Part 1: $r^E(m,t) > 0 \Rightarrow r^Q(m,t) > 0$. Suppose by contradiction that $r^Q(m,t) = 0$. Then $b(m,t) = 1$, meaning that $W_t^Q(m,t)$ attains maximum among all continuation payoffs (feasible or not). The initial assumption $r^Q(m,t) = 0$ then means that either $W_t^Q(\varnothing)$, or $W_t^Q(m,s)$ for some m and $s > t$ attain maximum, since one of these options should be more appealing to the quack than report (m,t). These payoffs, however, cannot attain maximum, since $b_t < 1$.

Part 2: $r^Q(m,t) > 0 \Rightarrow r^E(m,t) > 0$. Suppose that $r^E(m,t) = 0$. Since $r^Q(m,t) > 0$, we have $b(m,t) = 0$, and hence $W_t^Q(m,t)$ attains minimum among all continuation payoffs. However, since belief about the forecaster's type is a martingale from the observer's point of view, we have either $b(-m,t) > 0$ or $b_{t+1} > 0$. Thus at least one of these strategies (reporting $-m$ or staying silent at t) strictly dominates the strategy of reporting (m,t) for the quack, so $r^Q(m,t) = 0$.

Similarly, one can show that $r^Q(G,s) + r^Q(B,s) = 1$ if and only if $r^E(G,s) + r^E(B,s) = 1$. Indeed, if for some t we have $r^E(G,t) + r^E(B,t) = 1$ and $r^Q(G,t) + r^Q(B,t) < 1$, then not making a report by t grants quack a continuation payoff of zero, while by martingale property of belief there exists $m \in \{G,B\}$ such that $b(m,t) > 0$, and therefore report (m,t) dominates the strategy of staying silent. Similarly, if $r^E(G,t) + r^E(B,t) < 1$ and $r^Q(G,t) + r^Q(B,t) = 1$, then not making a report by t yields the maximal continuation payoff, while again by the martingale property making at least some report gives strictly less in expectation. $\square$

Before we proceed, it is useful to introduce some new pieces of notation which come in handy for further proofs. The expert's report probabilities can be rewritten as

$$\begin{aligned} r^E(m,t) = \mathbb{E}_\eta\left[r_\eta^E(m,t)\right] &= \frac{\tilde{p}_0 \cdot z_{t,G} \cdot r_G^E(m,t) + (1-\tilde{p}_0) \cdot z_{t,B} \cdot r_B^E(m,t) + z_{t,\varnothing} \cdot r_\varnothing^E(m,t)}{\tilde{p}_0 \cdot z_{t,G} + (1-\tilde{p}_0) \cdot z_{t,B} + z_{t,\varnothing}}, \\ \mathbb{E}_\eta[r_\eta^E(m,t)|\omega] &= \frac{\pi \cdot z_{t,\omega} \cdot r_\omega^E(m,t) + (1-\pi) \cdot z_{t,-\omega} \cdot r_{-\omega}^E(m,t) + z_{t,\varnothing} \cdot r_\varnothing^E(m,t)}{\pi \cdot z_{t,\omega} + (1-\pi) \cdot z_{t,-\omega} + z_{t,\varnothing}}, \end{aligned} \tag{36}$$

where $\tilde{p}_0 = p_0\pi + (1-p_0)(1-\pi)$, and $z_{t,\eta} = P\left\{\eta_t = \eta, \mu_{t-1} = \varnothing | \eta^* = \eta\right\}$ for $\eta \in \{\varnothing, G, B\}$. I.e.,

$z_{t,\eta}$ is the probability that the expert has information η at time t and has not made a report prior to t, conditional on expert's signal realization being $\eta^* = \eta$ (or unconditional if $\eta = \varnothing$). It can be expressed recursively as

$$\begin{aligned} z_{t,\eta} &= z_{t-1,\eta} \cdot \left(1 - \sum_m r^E_\eta(m, t-1)\right) + z_{t-1,\varnothing} \cdot \lambda(t) \cdot \left(1 - \sum_m r^E_\varnothing(m, t-1)\right), \\ z_{t,\varnothing} &= z_{t-1,\varnothing} \cdot (1 - \lambda(t)) \cdot \left(1 - \sum_m r^E_\varnothing(m, t-1)\right), \end{aligned} \tag{37}$$

with $z_{0,G} = z_{0,B} = 0$ and $z_{0,\varnothing} = 1$. In any symmetric equilibrium we have $z_{t,G} = z_{t,B} \equiv z_t$, so the expectations above transform into

$$r^E(m,t) = \mathbb{E}_\eta[r^E_\eta(m,t)] = z_t \left(\tilde{p}_0 r^E_G(m,t) + (1-\tilde{p}_0) r^E_B(m,t)\right) + (1 - z_t) r^E_\varnothing(m,t), \tag{38}$$

$$\mathbb{E}_\eta[r^E_\eta(m,t)|\omega] = z_t \left(\pi r^E_\omega(m,t) + (1-\pi) r^E_{-\omega}(m,t)\right) + (1 - z_t) r^E_\varnothing(m,t), \tag{39}$$

where $z_t = \frac{z_t}{z_t + z_{t,\varnothing}}$ and $1 - z_t = \frac{z_{t,\varnothing}}{z_t + z_{t,\varnothing}}$.

Given the strategies, we also define the *likelihood ratio of reports* as

$$g(m,t) := \ln\left(\beta(m,t)\right) - \ln\left(\beta_{t-1}\right) = \ln\left(\frac{r^E(m,t)}{r^Q(m,t)}\right),$$

with $\pm\infty$ being admissible values. This ratio summarizes the information about the forecaster's type contained in report (m,t).

Proof of Proposition 2. The proof is valid for all $\pi \in (\frac{1}{2}, 1]$. We begin with a useful observation:

$$W^E_{t,\varnothing}(m,t) = W^Q_t(m,t) \ \text{ for } m \in \{G, B\}. \tag{40}$$

If the expert reports (m,t) before observing a private signal, his continuation payoff coincides with that of the quack, since they possess the same private information at any such history.

Note further that the existence of a Godwin point $\bar{t} = \min\{t \in \mathcal{T} \mid V^E_{t,\varnothing} = V^Q_t\}$ is trivial since the required equality is always satisfied for the last point of $\mathcal{S}$. To see this, observe that any report (m,t) for $t > t_{|\mathcal{S}|}$ yields zero reputation for the rest of the game due to assumption (OP), and is therefore weakly dominated for any type of the forecaster by staying silent. At the same time, staying silent yields the same time-t expected payoff to the uninformed (as of time t) expert as it does to the quack, since they have the same information. This together with (40) gives the result.

Most of the remaining proof is devoted to showing that $V^E_{t,\varnothing} = V^Q_t$ for some t implies babbling in all further times. This is established in a series of claims. The second part of the proposition is then easily shown by contradiction.

As a starting point, we show that $W^E_{t,\eta}(m,t) = W^Q_t(m,t)$ for any $m \in \{G,B\}$, any $\eta \in \{G,B\}$ and any $t \in \mathcal{S}^m$ such that $t > \bar{t}$. Suppose the converse – there exist m, t and η such that $W^E_{t,\eta}(m,t) \neq W^Q_t(m,t)$. Then (ML) and (SY) imply that there can be three cases:

Case 1: $r^E_G(G,t) = r^E_B(B,t) \geqslant r^E_G(B,t) = r^E_B(G,t) > 0$.

In this case $W^E_{t,G}(G,t) = W^E_{t,G}(B,t)$ and, by Proposition 1, $V^Q_t = W^Q_t(G,t) = W^Q_t(B,t)$. Therefore, $W^E_{t,G}(G,t) - W^Q_t(G,t) = W^E_{t,G}(B,t) - W^Q_t(B,t)$, which reduces to the equality of differences in terminal reputation:

$$w^c\left(\beta^G(G,t)\right) - w^c\left(\beta^B(G,t)\right) = -w^c\left(\beta^B(B,t)\right) + w^c\left(\beta^G(B,t)\right). \tag{41}$$

If $r^E_G(G,t) > r^E_B(G,t)$ then, by (39) and the expression for $b^\omega(m,t)$, the LHS of (41) is weakly positive. However, due to (SY) we then have that $r^E_G(B,t) < r^E_B(B,t)$, so the RHS is weakly negative. The converse also holds, which leaves us with the conclusion that for (41) to be satisfied, its both sides must be equal to zero. Therefore, $\beta^G(m,t) = \beta^B(m,t)$ for any $m \in \{G,B\}$, which implies $W^E_{t,\eta}(m,t) = W^Q_t(m,t) = V^Q_t$ for any $\eta \in \{G,B\}$ and any $m \in \{G,B\}$, – a contradiction.

Case 2: $r^E_G(G,t) = r^E_B(B,t) > 0 = r^E_G(B,t) = r^E_B(G,t)$.

As $r^E_G(G,t) > r^E_B(G,t)$ and $r^E_B(B,t) > r^E_B(G,t)$, we have that $V^E_{t,G} = W^E_{t,G}(G,t) > W^Q_t(G,t) = V^Q_t$ and, analogously, $V^E_{t,B} > V^Q_t$. Next, note that $V^E_{\bar{t},\varnothing}$ is, for all $t > \bar{t}$, bounded below by

$$\sum_{s=\bar{t}}^{t-1} w\left(\beta_s\right) + P\{t^* \leqslant t \,|\, t^* > \bar{t}\} \cdot \left(\tilde{p}_0 \cdot V^E_{t,G} + (1-\tilde{p}_0) \cdot V^E_{t,B}\right) + P\{t^* > t \,|\, t^* > \bar{t}\} \cdot V^E_{t,\varnothing},$$

which is the value of not making a report from $\bar{t}$ until at least t. By (40) we have $V^E_{t,\varnothing} \geqslant W^E_{t,\varnothing}(m,t) = W^Q_t(m,t)$. Second, we have shown that $V^E_{t,\eta} > V^Q_t$. Therefore,

$$V^E_{\bar{t},\varnothing} > \sum_{s=\bar{t}}^{t-1} w\left(\beta_s\right) + W^Q_t(m,t) = V^Q_{\bar{t}},$$

which gives us a contradiction with the definition of $\bar{t}$.

Case 3: $r_G^E(G,t) = r_B^E(B,t) = r_G^E(B,t) = r_B^E(G,t) = 0$ and $r_\varnothing^E(m,t) > 0$ for some m.

$r_G^E(G,t) = r_G^E(B,t) = r_B^E(G,t) = r_B^E(B,t)$ automatically implies $W_{t,\eta}^E(m,t) = W_t^Q(m,t)$ for any $\eta \in \{G,B\}$ and any $m \in \{G,B\}$, which gives us a contradiction with the initial assumption.

Next we show that $W_{t,\eta}^E(m,t) = W_t^Q(m,t)$ for all $\eta, m \in \{G,B\}$ with $t \in \mathcal{S}^m$ implies that report (m,t) is babbling. Without loss of generality assume $\eta = G$. Expanding the equality, we see that

$$0 = W_{t,G}^E(m,t) - W_t^Q(m,t) = \frac{p_0 \cdot (1-p_0) \cdot (2\pi - 1)}{\tilde{p}_0} \cdot \left(w^c\left(\beta^G(m,t)\right) - w^c\left(\beta^B(m,t)\right)\right),$$

and therefore $\beta^G(m,t) = \beta^B(m,t)$. It further implies that (7) reduces to (10). In other words, it follows that reputation should not be affected by the revelation of state after any time-t report.

To conclude that only babbling is possible after $\bar{t}$ we are left to show that (9) holds for all (m,s) with $s > \bar{t}$. Condition (9) is equivalent to $g(m,s) = 0$. Three cases are possible (since we have shown in the proof of Proposition 1 that $r^\gamma(G,s) + r^\gamma(B,s) = 1$ cannot be the case for exactly one γ).

Case 1: $s = \max\{t \in \mathcal{S} \,|\, t > \bar{t}\}$ and $r^\gamma(G,s) + r^\gamma(B,s) = 1$ for any $\gamma \in \{S,Q\}$.

If m is the only report made at s then $r_\eta^\gamma(m,s) = 1$ for all γ, η, which implies $g(m,s) = 0$. If both reports are made on path at s, then by the same logic $r_\eta^\gamma(G,s) + r_\eta^\gamma(B,s) = 1$, and if $g(m,s) \neq 0$ for some m then the report with higher $g(m,s)$ is strictly preferred by either forecaster, contradicting that both reports occur on path.

Case 2: $s = \max\{t \in \mathcal{S} \,|\, t > \bar{t}\}$ and $r^\gamma(G,s) + r^\gamma(B,s) < 1$ for $\gamma \in \{S,Q\}$.

If m is the only report made at s and $g(m,s) \neq 0$ then a quack has strict preference between report (m,s) and staying silent at s (because in either case he gets a degenerate lottery at T, since (10) is satisfied). This strict preference cannot occur in equilibrium, thus $g(m,s) = 0$. If both reports are made on path at s then we can combine the two indifference arguments above to obtain that $g(G,s) = g(B,s)$ and, consequently, $g(m,s) = 0$ for all $m \in \{G,B\}$.

Case 3: $s < \max\{t \in \mathcal{S} \,|\, t > \bar{t}\}$.

From the previous case we know that $g(m,s) = 0$ for any on-path m at $s = \max\{t \in \mathcal{S} \,|\, t > \bar{t}\}$. We can iterate backwards from there as follows. If $s-1 \in \mathcal{S}$ then a quack should be indifferent between making an on-path report at $s-1$ and at s, which can only happen if $g(m,s-1) = 0$, because $g(m,s)=0$ and (10) is satisfied for both reports. Iterating backwards we establish the claim for all $t>\bar{t}$. If some of these periods are not in $\mathcal{S}$ then they can be skipped because beliefs do not change at such periods.

All of the above proves that only babbling is possible after the Godwin point.

We are left to show the second part of the proposition. First, suppose there exist m and $t<\bar{t}$ such that $r^E_{\varnothing}(m,t)>0$. Then $V^E_{t,\varnothing}=W^E_{t,\varnothing}(m,t)=W^Q_t(m,t)=V^Q_t$, where the second equality follows from (40). Thus $t\geqslant\bar{t}$ by definition of $\bar{t}$ – a contradiction.

Now suppose there exists $t\leqslant\bar{t}$ such that $r^E_\eta(-\eta,t)>0$ for some $\eta\in\{G,B\}$. As shown before, it implies $V^E_{t,\eta}=W^E_{t,\eta}(m,t)=W^Q_t(m,t)=V^Q_t$ for all $\eta,m\in\{G,B\}$. Suppose first that $t=\bar{t}$. Then as $r^E_{\varnothing}(m,t_{|\mathcal{S}|-1})=0$ for $m\in\{G,B\}$, we have

$$V^E_{t_{|\mathcal{S}|-1},\varnothing}=w\left(\beta_{t_{|\mathcal{S}|-1}}\right)+P\{t^*\leqslant\bar{t}\,|\,t^*>t_{|\mathcal{S}|-1}\}\cdot\left(\tilde{p}_0\cdot V^E_{\bar{t},G}+(1-\tilde{p}_0)\cdot V^E_{\bar{t},B}\right)+P\{t^*>\bar{t}\,|\,t^*>t_{|\mathcal{S}|-1}\}\cdot V^E_{\bar{t},\varnothing}.$$

As $V^E_{\bar{t},\varnothing}=V^Q_{\bar{t}}$ and $V^E_{\bar{t},\eta}=V^Q_{\bar{t}}$, the above expression reduces to $V^E_{t_{|\mathcal{S}|-1},\varnothing}=w\left(\beta_{t_{|\mathcal{S}|-1}}\right)+V^Q_{\bar{t}}=V^Q_{t_{|\mathcal{S}|-1}}$, which constitutes a contradiction with the definition of the Godwin point. One can similarly show that $r^E_\eta(\eta,\bar{t})>0$, as otherwise $V^E_{\bar{t},\eta}=W^E_{\bar{t},\eta}(m,\bar{t})=W^Q_t(m,\bar{t})=V^Q_{\bar{t}}$ for all $\eta\in\{G,B\}$ and all $m\in\{G,B\}$, which leads to the same contradiction with the definition of the Godwin point as above. Finally, if $t<\bar{t}$ then, as $r^E_\eta(-\eta,\bar{t})=0$ and $r^E_\eta(\eta,\bar{t})>0$ imply $V^E_{\bar{t},\eta}>V^Q_{\bar{t}}$, a competent forecaster who has received a signal by period t can postpone his report until $\bar{t}$ and receive strictly more than the quack which contradicts $V^E_{t,\eta}=V^Q_t$ implied by $r^E_\eta(-\eta,t)>0$.

We are left to show that the expert never wants to conceal his private signal. Assume $r^E_\eta(\eta,\bar{t})<1$. Then thet expert must weakly prefer to conceal his private signal than to report it. In the first case the expert receives exactly $V^Q_{\bar{t}}$, while in the latter he gets $V^E_{\bar{t},\eta}>V^Q_{\bar{t}}$, – a contradiction. □

Proof of Proposition 3. The proof is valid for all $\pi\in\left(\frac{1}{2},1\right]$. Let $\{r^\gamma_\eta(m,t)\}$ be an equilibrium strategy profile. Consider a new strategy profile $\{\tilde{r}^\gamma_\eta(m,t)\}$ such that $\tilde{r}^E_{t,\eta}(m,t)=r^E_{t,\eta}(m,t)$, $\tilde{r}^Q_t(m,t)=r^Q_t(m,t)$ for all $t\leqslant\bar{t}$ and $\tilde{r}^E_{t,\eta}(m,t)=\tilde{r}^Q_t(m,t)=0$ for all $t>\bar{t}$. As strategies coincide on $\tilde{\mathcal{S}}$ and all reports (m,t) with $t>\bar{t}$ are babbling in the original equilibrium, the following are true:

1. beliefs $b(m,t)$ and $b^\omega(m,t)$ induced by the two strategy profiles coincide for all $\omega,m\in\{G,B\}$, $t\in\tilde{\mathcal{S}}$;

2. belief sequences b_t induced by the two strategy profiles coincide for all $t \in \mathcal{T}$.

The latter statement also exploits the fact that $\mathcal{S} \backslash \tilde{\mathcal{S}}$ is nonempty (otherwise the proposition statement trivially holds), so it must be that $r^Q(G, \bar{t}) + r^Q(B, \bar{t}) < 1$ and $r^E(G, \bar{t}) + r^E(B, \bar{t}) < 1$.

The first statement above implies that any report (m, t) with $t \leqslant \bar{t}$ yields the same payoff under either strategy profile. The second statement states that reporting nothing in any period yields the same payoffs as well. Strategy of reporting nothing yields the same payoff under $\{\tilde{r}^\gamma_\eta(m,t)\}$ as any report (m, t) with $t > \bar{t}$ under $\{r^\gamma_\eta(m,t)\}$, since all such reports are babbling. Finally, any report (m, t) with $t \notin \mathcal{S}$ yields the same payoff under either strategy profile due to (OP).

Everything said above directly implies that if $r^\gamma_\eta(m,t)$ is a best response for type-γ forecaster to strategy profile $\{r^\gamma_\eta(m,t)\}$ then $\tilde{r}^\gamma_\eta(m,t)$ is a best response for him to strategy profile $\{\tilde{r}^\gamma_\eta(m,t)\}$ and yields the same payoff. $\square$

Before proceeding to the proof of Theorem 1, we provide some expressions for belief updating that will be useful in further proofs. Using Proposition 2 and the notion of z_t introduced earlier in this Appendix, we can rewrite the expressions for (4) and (6) in a more explicit form. Proposition 2 implies that for all $t < \bar{t}$ we have $r^E_G(B,t) = r^E_B(G,t) = 0$. Therefore, (38) and (39) together imply that for all $t \in \mathcal{S}$ we have

$$\begin{aligned} \beta(G,t) &= \beta_{t-1} \cdot \frac{r^E(G,t)}{r^Q(G,t)} = \beta_{t-1} \cdot \frac{\tilde{p}_0 z_t r^E_G(G,t) + (1-z_t) r^E_\varnothing(G,t)}{r^Q(G,t)}, \\ \beta(B,t) &= \beta_{t-1} \cdot \frac{r^E(B,t)}{r^Q(B,t)} = \beta_{t-1} \cdot \frac{(1-\tilde{p}_0)\, z_t r^E_B(B,t) + (1-z_t) r^E_\varnothing(B,t)}{r^Q(B,t)}, \end{aligned} \tag{42}$$

as well as

$$\begin{aligned} \beta^G(G,t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_\eta[r^E_\eta(G,t)|G]}{r^Q(G,t)} = \beta_{t-1} \cdot \frac{\pi z_t r^E_G(G,t) + (1-z_t) r^E_\varnothing(G,t)}{r^Q(G,t)}, \\ \beta^B(B,t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_\eta[r^E_\eta(B,t)|B]}{r^Q(B,t)} = \beta_{t-1} \cdot \frac{\pi z_t r^E_B(B,t) + (1-z_t) r^E_\varnothing(B,t)}{r^Q(B,t)}, \\ \beta^B(G,t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_\eta[r^E_\eta(G,t)|B]}{r^Q(G,t)} = \beta_{t-1} \cdot \frac{(1-\pi) z_t r^E_G(G,t) + (1-z_t) r^E_\varnothing(G,t)}{r^Q(G,t)}, \\ \beta^G(B,t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_\eta[r^E_\eta(B,t)|G]}{r^Q(B,t)} = \beta_{t-1} \cdot \frac{(1-\pi) z_t r^E_B(B,t) + (1-z_t) r^E_\varnothing(B,t)}{r^Q(B,t)}, \end{aligned} \tag{43}$$

and It is also worth remembering that $r^E_\varnothing(m,t) = 0$ for any $t < \bar{t}$.

In case no report was made in period $t < \bar{t}$, the belief is updated as

$$\beta_t = \beta_{t-1} \cdot \frac{1 - r^E(G,t) - r^E(B,t)}{1 - r^Q(G,t) - r^Q(B,t)} = \beta_{t-1} \cdot \frac{1 - z_t r^E_G(G,t)}{1 - r^Q(G,t) - r^Q(B,t)}, \tag{44}$$

while the analogous expression for $t = \bar{t}$ is given by

$$\beta_{\bar{t}} = \beta_{t_{|\mathcal{S}|-1}} \cdot \frac{1 - r^E(G,\bar{t}) - r^E(B,\bar{t})}{1 - r^Q(G,\bar{t}) - r^Q(B,\bar{t})} = \beta_{t_{|\mathcal{S}|-1}} \cdot \frac{(1 - z_{\bar{t}}) \cdot \left(1 - r^E_{\varnothing}(G,\bar{t}) - r^E_{\varnothing}(B,\bar{t})\right)}{1 - r^Q(G,\bar{t}) - r^Q(B,\bar{t})}. \tag{45}$$

In (44) we use the fact that $r^E_G(G,t) = r^E_B(B,t)$ due to (SY), and in (45) we use that $r^E_G(G,\bar{t}) = r^E_B(B,\bar{t}) = 1$ by Proposition 2.

What follows is the proof of the main result, Theorem 1. To avoid duplicating the arguments, we merge it with the proof of Propositions 7 and 9.

Proof of Theorem 1 and Propositions 7 and 9. . First, recall that Propositions 1, 2 and 3 are true for all $\pi \in \left(\frac{1}{2}, 1\right]$, and so can be employed in this proof. Further, note that in all babbling periods t we have $i(m,t) = 0$, $b(m,t) = b_{t-1}$ for $m \in \{G, B\}$, and b_t stays on a constant level. Together with Propositions 2 and 3 this means that it is enough to show the statement of the Theorem for informative equilibria. The proof is separated into several steps.

Step 1. We start by showing that $\varDelta w_\eta(m,t)$, which is defined as

$$\varDelta w_\eta(m,t) := w^c\left(\beta^\eta(m,t)\right) - w^c\left(\beta^{-\eta}(m,t)\right),$$

is a weakly decreasing function of t on $\mathcal{S}$ given $m = \eta$ (note that $\varDelta w_G(m,t) = -\varDelta w_B(m,t)$). Suppose the expert has private information $\eta = G$ at time t, but has not yet made any report. He chooses a report (m,τ) with $\tau \geqslant t$ which maximizes $W^E_{t,G}(m,\tau)$, where "making no report" is also an available option. Expanding $W^E_{t,G}(m,\tau)$, we get the following expression:

$$\sum_{s=t}^{\tau-1} w\left(\beta_s\right) + \sum_{s=\tau}^{T-1} w\left(\beta(m,\tau)\right) + \frac{\pi p_0}{\tilde{p}_0} w^c\left(\beta^G(m,\tau)\right) + \left(1 - \frac{\pi p_0}{\tilde{p}_0}\right) w^c\left(\beta^B(m,\tau)\right),$$

where $\tilde{p}_0 = p_0\pi + (1-p_0)(1-\pi)$.

A quack is indifferent between all such reports at time t. His continuation value $W^Q_t(m,\tau)$ can similarly be written as

$$\sum_{s=t}^{\tau-1} w\left(\beta_s\right) + \sum_{s=\tau}^{T-1} w\left(\beta(m,\tau)\right) + p_0 w^c\left(\beta^G(m,\tau)\right) + (1-p_0)\, w^c\left(\beta^B(m,\tau)\right).$$

Given that the latter expression is constant over all (m,τ), the optimization problem of an expert with $\eta = G$ becomes equivalent to maximizing the difference $\varDelta w_G(m,\tau)$ over all $\tau \in \{\mathcal{S} | \tau \geqslant t\}$ and $m \in \{G, B\}$.

Similarly, an expert who has observed signal B chooses report (m,τ) which maximizes $\varDelta w_B(m,\tau)$. Propositions 1, 2 and 3 imply that $\mathcal{S}=\{t\in\mathcal{T}\,|\,r^E_\eta(\eta,t)>0\}$ for any $\eta\in\{G,B\}$, so since $t\in\mathcal{S}$, it must be that (G,t) maximizes $\varDelta w_G(m,\tau)$ and (B,t) maximizes $\varDelta w_B(m,\tau)$ across all (m,τ) with $\tau\in\{\mathcal{S}\,|\,\tau\geqslant t\}$. Therefore, $\varDelta w_\eta(\eta,t)$ must be a weakly decreasing function of t on $\mathcal{S}$.

Step 2. The second step of the proof consists in showing that for $\eta\in\{G,B\}$, $\varDelta w_\eta(\eta,t)$ is weakly decreasing on $\mathcal{S}\backslash\{\bar{t}\}$ if and only if $\beta^\eta(\eta,t)$ is weakly decreasing on $\mathcal{S}\backslash\{\bar{t}\}$ (if an equilibrium is reticent then $\varDelta w_\eta(\eta,t)$ is weakly decreasing on $\mathcal{S}$ if and only if $\beta^\eta(\eta,t)$ is weakly decreasing on $\mathcal{S}$). We demonstrate it for all cases stated in Theorem 1, Proposition 7 and Proposition 9 separately.

Case 1: $\pi=1$.

This case is obvious, as then $w\left(\beta^{-\eta}(\eta,t)\right)=0$ for any $\eta\in\{G,B\}$, and $w(\cdot)$ is a strictly increasing function.

Case 2: $w^c(\cdot)$ is convex and $\pi<1$.

Note that since $r^E_\varnothing(m,t)=0$ for all $t\in\mathcal{S}\backslash\{\bar{t}\}$ (or for all $t\in\mathcal{S}$ if an equilibrium is reticent) from (43) we have

$$\beta^{-\eta}(\eta,t)=\frac{1-\pi}{\pi}\cdot\beta^\eta(\eta,t),$$

where $\frac{1-\pi}{\pi}\in(0,1)$ because $\pi>\frac{1}{2}$. Take any $\tau_1>\tau_2$ with $\tau_1,\tau_2\in\mathcal{S}$. Then if $\beta^\eta(\eta,\tau_1)=x_1>x_2=\beta^\eta(\eta,\tau_2)$ we have

$$w^c\left(x_1\right)-w^c\left(\frac{1-\pi}{\pi}x_1\right)\geqslant w^c\left(x_2\right)-w^c\left(x_2-x_1+\frac{1-\pi}{\pi}x_1\right)>w^c\left(x_2\right)-w^c\left(\frac{1-\pi}{\pi}x_2\right),$$

where the first inequality follows from convexity of $w(\beta)$, and the second is valid because $w(\beta)$ is strictly increasing.

Case 3: $\pi>\frac{\bar{d}}{\underline{d}+\bar{d}}$ and $\frac{dw^c(\beta)}{d\beta}\in[\underline{d},\bar{d}]$.

Similarly to the previous case take any $\tau_1,\tau_2\in\mathcal{S}$ and let $x_1:=\beta^\eta(\eta,\tau_1)$, $x_2:=\beta^\eta(\eta,\tau_2)$. Suppose $w^c\left(x_2\right)-w^c\left(\frac{1-\pi}{\pi}x_2\right)>w^c\left(x_1\right)-w^c\left(\frac{1-\pi}{\pi}x_1\right)$. Then

$$\begin{aligned}0&<\left(w^c\left(x_2\right)-w^c\left(\frac{1-\pi}{\pi}x_2\right)\right)-\left(w^c\left(x_1\right)-w^c\left(\frac{1-\pi}{\pi}x_1\right)\right)<\\&<\bar{d}\cdot(x_2-x_1)-\underline{d}\cdot\left(\frac{1-\pi}{\pi}x_2-\frac{1-\pi}{\pi}x_1\right)<(x_2-x_1)\cdot\left(\bar{d}-\underline{d}\cdot\frac{1-\pi}{\pi}\right).\end{aligned}$$

As $\left(\bar{d}-\underline{d}\cdot\frac{1-\pi}{\pi}\right)>0$, we must have $x_2>x_1$.

Conversely, if $x_2 > x_1$ then

$$0 < (x_2 - x_1) \cdot \left(\underline{d} - \bar{d} \cdot \frac{1-\pi}{\pi}\right) < \left(w^c(x_2) - w^c\left(\frac{1-\pi}{\pi} x_2\right)\right) - \left(w^c(x_1) - w^c\left(\frac{1-\pi}{\pi} x_1\right)\right),$$

which grants the result.

Step 3. We next show that whenever $|\mathcal{S}| \geqslant 3$ and an equilibrium on $\mathcal{S}$ exists, it must be that $b_{t_1} \geqslant b(m, t_1)$ for any $m \in \{G, B\}$ (alternatively, if equilibrium is reticent, then this claim is valid for any $\mathcal{S}$ with $|\mathcal{S}| \geqslant 2$). Assume there exists $m \in \{G, B\}$ such that $b_{t_1} < b(m, t_1)$. Quack's value from report m at t_1 and t_2 are equal to

$$W_{t_1}^Q(m, t_1) = (T - t_1) \cdot w(\beta(m, t_1)) + p_0 w^c\left(\beta^G(m, t_1)\right) + (1 - p_0)\, w^c\left(\beta^B(m, t_1)\right),$$
$$W_{t_1}^Q(m, t_2) = (t_2 - t_1) \cdot w(\beta_{t_1}) + (T - t_2) \cdot w(\beta(m, t_2)) + p_0 w^c\left(\beta^G(m, t_2)\right) + (1 - p_0) w^c\left(\beta^B(m, t_2)\right).$$

As $w(\cdot)$ is strictly increasing, and $b_{t_1} < b(m, t_1)$, $W_{t_1}^Q(m, t_1) = W_{t_1}^Q(m, t_2)$ implies

$$(T - t_2) \cdot w(\beta(m, t_1)) + p_0 w^c\left(\beta^G(m, t_1)\right) + (1 - p_0)\, w^c\left(\beta^B(m, t_1)\right) <$$
$$(T - t_2) \cdot w(\beta(m, t_2)) + p_0 w^c\left(\beta^G(m, t_2)\right) + (1 - p_0) w^c\left(\beta^B(m, t_2)\right).$$

Consequently, it must be that either $\beta(m, t_1) < \beta(m, t_2)$, or $\beta^m(m, t_1) < \beta^m(m, t_2)$ or $\beta^{-m}(m, t_1) < \beta^{-m}(m, t_2)$. However, (42) and (43) imply that both $\beta^G(m, t)$ and $\beta^B(m, t)$ differ from $\beta(m, t)$ by a constant factor for any $t \in \mathcal{S} \backslash \{\bar{t}\}$ (since $r_\varnothing^E(m, t) = 0$), so the three inequalities are equivalent. Therefore, $\beta^m(m, t_1) < \beta^m(m, t_2)$, which contradicts $\beta^m(m, t)$ being decreasing on $\mathcal{S} \backslash \{\bar{t}\}$. In reticent equilibria $r_\varnothing^E(m, \bar{t}) = 0$, therefore the claim extends to $\bar{t}$ as well.

Step 4. We finally show how the claim in the theorem can be obtained from the previous steps. We have shown that $b_{t_1} \geqslant b(m, t_1)$ for any $m \in \{G, B\}$. Consequently, as b_t is a martingale, we have that $b_{t_1} \geqslant b_0$ and $b(m, t_1) \leqslant b_0$ for at least one $m \in \{G, B\}$. As $b(m, t_1) \leqslant b_{t_1}$ for $m \in \{G, B\}$, we must have that either $b(m, t_2) \leqslant b(m, t_1)$, or $b^m(m, t_2) \leqslant b^m(m, t_1)$ or $b^{-m}(m, t_2) \leqslant b^{-m}(m, t_1)$ to make the quack indifferent between reports (m, t_1) and (m, t_2). Again, (42) and (43) imply that all three inequalities are equivalent, so all three have to hold. The fact that b_t is a martingale together with resulting inequalities $b_{t_1} \geqslant b(m, t_1) \geqslant b(m, t_2)$ for $m \in \{G, B\}$ imply $b_{t_2} \geqslant b_{t_1}$. Iterating this argument further, we achieve that $b(m, t) \leqslant b_t$ and b_t is increasing in t on $\mathcal{S} \backslash \{\bar{t}\}$ (on whole $\mathcal{S}$ if equilibrium is reticent).

The above proves the second and the third parts of Theorem 1 and Propositions 7 and 9.[88] It remains to show the first part. Note that, by the same inductive reasoning as above, if $b_{t_1} > b(m, t_1)$ then $b_{t_{|\mathcal{S}|-1}} > b(m, t_{|\mathcal{S}|-1})$. Consequently, it is possible to show that $b^m(m, \bar{t}) < b^m(m, t_{|\mathcal{S}|-1})$. Indeed, suppose the converse. Then to make the quack indifferent between reporting m at $t_{|\mathcal{S}|-1}$ and $\bar{t}$, we must have $b^{-m}(m, \bar{t}) < b^{-m}(m, t_{|\mathcal{S}|-1})$. But then

$$w^c\left(\beta^m(m, t_{|\mathcal{S}|-1})\right) - w^c\left(\beta^{-m}(m, t_{|\mathcal{S}|-1})\right) < w^c\left(\beta^m(m, \bar{t})\right) - w^c\left(\beta^{-m}(m, \bar{t})\right),$$

which contradicts the fact that $\Delta w_\eta(m, \tau)$ is weakly decreasing in t on $\mathcal{S}$ for $m = \eta$.

Finally, remember that for all $t \in \mathcal{S}\backslash\{\bar{t}\}$ we have

$$|i(m,t)| = \ln\left(\frac{1 + \beta^m(m,t)}{1 + \beta^{-m}(m,t)}\right) = \ln\left(\frac{1 + \beta^m(m,t)}{1 + \frac{1-\pi}{\pi}\beta^m(m,t)}\right), \tag{46}$$

which is then a decreasing function of t on $\mathcal{S}\backslash\{\bar{t}\}$ as well because $\ln(1+x) - \ln\left(1 + \frac{1-\pi}{\pi}x\right)$ is an increasing function of x. For the last two points of $\mathcal{S}$ we have

$$|i(m, t_{|\mathcal{S}|-1})| - |i(m, \bar{t})| = \ln\left(\frac{1 + \beta^m(m, t_{|\mathcal{S}|-1})}{1 + \frac{1-\pi}{\pi}\beta^m(m, t_{|\mathcal{S}|-1})}\right) - \ln\left(\frac{1 + \beta^m(m, \bar{t})}{1 + \frac{1-\pi}{\pi}\beta^m(m, \bar{t}) + \frac{2\pi-1}{\pi} \cdot \frac{(1-z_{\bar{t}})r^E_{\varnothing}(m,\bar{t})}{r^Q(m,\bar{t})}}\right) > 0,$$

where the last inequality follows from $\beta^m(m, t_{|\mathcal{S}|-1}) > \beta^m(m, \bar{t})$ and the fact that $\pi > \frac{1}{2}$. This concludes the proof of Theorem 1/Proposition 9 for general informative equilibria. □

We continue by presenting the proof of Proposition 6, which is a special case of Theorem 1/Proposition 9 for delay equilibria.

Proof of Proposition 6. Let $\{r^\gamma_\eta(m,t)\}$ constitute a delay equilibrium on $\mathcal{S}$. First, note that if $b_{t_1} = b(G, t_1) = b(B, t_1)$ then $b_t = b(m,t) = b_0$ for $t \in \mathcal{S}\backslash\{\bar{t}\}$. It further implies that $b^m(m,t)$ is constant on $\mathcal{S}\backslash\{\bar{t}\}$. Therefore, $i(m,t)$ is constant on $\mathcal{S}\backslash\{\bar{t}\}$ as well, as suggested by (46), so we get all three statements. If equilibrium is reticent then all these claims are valid for all $t \in \mathcal{S}$.

To show that $b_{t_1} = b(G, t_1) = b(B, t_1)$, proceed by contradiction. If there exists $m \in \{G, B\}$ such that $b_{t_1} > b(m, t_1)$ then $r^E_m(m,t) = 1$ for all $t \in \mathcal{S}\backslash\{t_{|\mathcal{S}|-1}\}$.[89] Due to (SY), the same applies to the other m as well. Further, if such m exists then, as shown above, $b^m(m, \bar{t}) < b^m(m, t_{|\mathcal{S}|-1})$,

[88] The statement that b_t is constant on $\mathcal{T}\backslash\mathcal{S}$ follows trivially from (4).

[89] The claim for all points except the two last ones follows from the fact that $b^m(m,t)$ is strictly decreasing in this case. Furthermore, remember that $r^E_m(m,\bar{t}) = 1$ for $m \in \{G, B\}$ by Proposition 2.

meaning that

$$w^c\left(\beta^m(m,t_{|\mathcal{S}|-1})\right)-w^c\left(\frac{1-\pi}{\pi}\beta^m(m,t_{|\mathcal{S}|-1})\right)>w^c\left(\beta^m(m,\bar{t})\right)-w^c\left(\frac{1-\pi}{\pi}\beta^m(m,\bar{t})\right)$$

It follows from the fact that $w^c(x_1)-w^c\left(\frac{1-\pi}{\pi}x_1\right)>w^c(x_2)-w^c\left(\frac{1-\pi}{\pi}x_2\right)$ if and only if $x_1>x_2$ whenever $\pi=1$ (which corresponds to Theorem 1) or any of the two conditions in Proposition 9 are satisfied. Finally, (43) implies that $\frac{1-\pi}{\pi}\beta^m(m,t_{|\mathcal{S}|-1})=\beta^{-m}(m,t_{|\mathcal{S}|-1})$ and $\frac{1-\pi}{\pi}\beta^m(m,\bar{t})<\beta^{-m}(m,\bar{t})$, which together give

$$w^c\left(\beta^m(m,t_{|\mathcal{S}|-1})\right)-w^c\left(\beta^{-m}(m,t_{|\mathcal{S}|-1})\right)>w^c\left(\beta^m(m,\bar{t})\right)-w^c\left(\beta^{-m}(m,\bar{t})\right).$$

The resulting inequality means that informed expert strictly prefers to report his private information at $t_{|\mathcal{S}|-1}$ rather than at $\bar{t}$. This is the last step towards the conclusion that if there exists $m\in\{G,B\}$ such that $b_{t_1}>b(m,t_1)$ then $r^E_m(m,t)=1$ for $m\in\{G,B\}$ and all $t\in\mathcal{S}$, which is a contradiction to $\{r^\gamma_\eta(m,t)\}$ constituting a *delay* equilibrium. □

Proving statements about equilibrium existence and properties requires showing some supplementary results first. We start with a lemma that shows that delay equilibria can effectively be considered as modifications of relay equilibria.

Lemma 5. *For any delay equilibrium on $\mathcal{S}$ with $|\mathcal{S}|\geqslant 3$ there exists a payoff-equivalent relay equilibrium, such that beliefs after the same histories coincide in the two equilibria.*

Proof. Assume that strategy profile $\{r^\gamma_\eta(m,t)\}$ constitutes a delay equilibrium on $\mathcal{S}$. Consider strategy profile $\{\tilde{r}^\gamma_\eta(m,t)\}$ such that

1. $\tilde{r}^E_\eta(m,\bar{t})=r^E_\eta(m,\bar{t})$ and $\tilde{r}^Q(m,\bar{t})=r^Q(m,\bar{t})$ for $\eta\in\{\varnothing,G,B\}$ and $m\in\{G,B\}$;
2. $\tilde{r}^E_\eta(m,t)=1$ for $m=\eta$, $\tilde{r}^E_\eta(m,t)=0$ for $m\neq\eta$, and $\tilde{r}^Q(m,t)=\frac{r^Q(m,t)}{r^E_m(m,t)}$ for all $t\in\mathcal{S}\backslash\{\bar{t}\}$.

By Proposition 6 a strategy profile constitutes a delay equilibrium on $\mathcal{S}$ with $|\mathcal{S}|\geqslant 3$ only if $b_t=b(G,t)=b(B,t)=b_0$ for all $t\in\mathcal{S}\backslash\{\bar{t}\}$. Therefore $r^Q(m,t)=r^E(m,t)$ for $m\in\{G,B\}$ and all $t\in\mathcal{S}\backslash\{\bar{t}\}$. Consequently, $\tilde{r}^Q(m,t)=\frac{r^Q(m,t)}{r^E_m(m,t)}<\frac{r^Q(m,t)}{r^E(m,t)}=1$, that is $\tilde{r}^E_\eta(m,t)=1$ is indeed a well-defined profile of strategies. Moreover, profile $\{\tilde{r}^\gamma_\eta(m,t)\}$ induces the same beliefs as profile $\{r^\gamma_\eta(m,t)\}$ after the same histories, and therefore also constitutes an equilibrium. At the same time, this equilibrium is a relay one because $r^E_G(G,m)=r^E_B(B,m)=1$. □

Next we proceed with describing which conditions are necessary for a given profile of strategies $\{r^{\gamma}_{\eta}(m,t)\}$ to constitute a *relay* equilibrium. We consider two sub-cases depending on whether not making a report by $\bar{t}$ is on equilibrium path.

Lemma 6. *Suppose that beliefs $\beta(m,t)$ and $\beta^{\omega}(m,t)$ for all $t \in \mathcal{S}$ are given by* (42) *and* (43) *respectively, while β_t is given by* (44) *for all $t < \bar{t}$. Moreover, let strategy profile $\{r^{\gamma}_{\eta}(m,t)\}$ be such that: (1) $r^{E}_{\eta}(\eta,t) = 1$ for all $\eta \in \{G,B\}$ and all $t \in \mathcal{S}$, and (2) $r_{\varnothing}(m,t) = 0$ for all $m \in \{G,B\}$ and all $t \in \mathcal{S}\backslash\{\bar{t}\}$.*

1. *Strategy profile $\{r^{\gamma}_{\eta}(m,t)\}$ with $r^{E}_{\varnothing}(G,\bar{t}) + r^{E}_{\varnothing}(B,\bar{t}) = 1$ constitutes a relay equilibrium on $\mathcal{S}$ only if*
$$\begin{aligned} &W^{Q}_{t_1}(m,t) = \bar{W} \text{ for all } t \in \mathcal{S} \text{ and } m \in \{G,B\} \text{ for some } \bar{W} \in \mathbb{R}_+, \\ &r^{Q}(G,\bar{t}) + r^{Q}(B,\bar{t}) = 1. \end{aligned} \tag{47}$$
Moreover, there exists at most one solution to this system, and if $w(\beta) = \beta^{\alpha}$ and $w^{c}(\beta) = \theta \cdot \beta^{\alpha}$ then this solution always exists.

2. *If* (45) *holds then strategy profile $\{r^{\gamma}_{\eta}(m,t)\}$ with $r^{E}_{\varnothing}(G,\bar{t}) + r^{E}_{\varnothing}(B,\bar{t}) < 1$ constitutes a relay equilibrium on $\mathcal{S}$ only if*
$$W^{Q}_{t_1}(m,t) = W^{Q}_{t_1}(\varnothing) = \bar{W} \text{ for all } t \in \mathcal{S} \text{ and } m \in \{G,B\} \text{ for some } \bar{W} \in \mathbb{R}_+. \tag{48}$$
Moreover, there exists at most one solution to this system, and if $w(\beta) = \beta^{\alpha}$ and $w^{c}(\beta) = \theta \cdot \beta^{\alpha}$ then this solution always exists.

Proof. By Proposition 1, a strategy profile constitutes an equilibrium only if $W^{Q}_{t_1}(m,t)$ is constant for all $t \in \mathcal{S}$ and $m \in \{G,B\}$. Additionally, if $r^{E}_{\varnothing}(G,\bar{t}) + r^{E}_{\varnothing}(B,\bar{t}) < 1$ – that is, not making a report by $\bar{t}$ is an on-path action – the value that the quack receives from making any report must be equal to value from making no report.

The proof of Proposition 1 argued that $r^{E}(G,\bar{t}) + r^{E}(B,\bar{t}) = 1$ implies $r^{Q}(G,\bar{t}) + r^{Q}(B,\bar{t}) = 1$. From Proposition 2 we know that $r^{E}_{\eta}(\eta,\bar{t}) = 1$, and therefore $r^{E}(G,\bar{t}) + r^{E}(B,\bar{t}) = 1$ is equivalent to $r^{E}_{\varnothing}(G,\bar{t}) + r^{E}_{\varnothing}(B,\bar{t}) = 1$. This completes the proof of the first parts of both statements.

To prove the uniqueness of solutions to respective systems, assume that there exist two different strategy profiles $\{r^{\gamma}_{\eta}(m,t)\}$, $\{\tilde{r}^{\gamma}_{\eta}(m,t)\}$, values of making a report $W^{Q}_{t}(m,\tau)$, $\tilde{W}^{Q}_{t}(m,\tau)$, and belief profiles b, $\tilde{b}$ that solve either system (47) or system (48). Then $r^{Q}(G,t_1) \neq \tilde{r}^{Q}(G,t_1)$, as otherwise equilibria coincide. Indeed, strategies $r^{E}_{\eta}(m,t) = \tilde{r}^{E}_{\eta}(m,t)$ for all $t \in \mathcal{S}$. Therefore, if $r^{Q}(G,t_1) =$

$\tilde{r}^Q(G,t_1)$ then $b(G,t_1) = \tilde{b}(G,t_1)$ and $b^\omega(G,t_1) = \tilde{b}^\omega(G,t_1)$, meaning that $W^Q_{t_1}(G,t_1) = \tilde{W}^Q_{t_1}(G,t_1)$. By the first two parts of the lemma, the quack's values $W^Q_{t_1}(m,t)$ should then coincide for all m and $t \in \mathcal{S}$, which implies $r^Q(m,t) = \tilde{r}^Q(m,t)$ – a contradiction.

Without loss, assume $r^Q(G,t_1) > \tilde{r}^Q(G,t_1)$. Then since $W^Q_{t_1}(G,t_1) = W^Q_{t_1}(B,t_1)$ and $\tilde{W}^Q_{t_1}(G,t_1) = \tilde{W}^Q_{t_1}(B,t_1)$, we must have $r^Q(B,t_1) > \tilde{r}^Q(B,t_1)$ as well. By (44) this implies that $b_{t_1} > \tilde{b}_{t_1}$. Consequently, $r^Q(G,t_2) > \tilde{r}^Q(G,t_2)$ and $r^Q(B,t_2) > \tilde{r}^Q(B,t_2)$ because $W^Q_{t_1}(m,t_1) = W^Q_{t_1}(m,t_2)$ and $\tilde{W}^Q_{t_1}(m,t_1) = \tilde{W}^Q_{t_1}(m,t_2)$ for $m \in \{G,B\}$. Iterating this logic further, we obtain that $r^Q(G,\bar{t}) + r^Q(B,\bar{t}) > \tilde{r}^Q(G,\bar{t}) + \tilde{r}^Q(B,\bar{t})$. In the context of the first part of the lemma (case $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) = 1$), it clearly violates $r^Q(G,\bar{t}) + r^Q(B,\bar{t}) = \tilde{r}^Q(G,\bar{t}) + \tilde{r}^Q(B,\bar{t}) = 1$. In the context of the second part, it implies $b_{\bar{t}} > \tilde{b}_{\bar{t}}$, and therefore $W^Q_{t_1}(\varnothing) > \tilde{W}^Q_{t_1}(\varnothing)$ because the payoff that the quack receives from staying silent is point-wise lower in the former equilibrium. At the same time, because $r^Q(G,t_1) > \tilde{r}^Q(G,t_1)$, we must have $W^Q_{t_1}(G,t_1) < \tilde{W}^Q_{t_1}(G,t_1)$. As in the second case $W^Q_{t_1}(\varnothing) = W^Q_{t_1}(G,t_1)$ and $\tilde{W}^Q_{t_1}(\varnothing) = \tilde{W}^Q_{t_1}(G,t_1)$, we arrive to a contradiction.

Finally, to prove existence of a solution for $w(\beta) = \beta^\alpha$ and $w^c(\beta) = \theta \cdot \beta^\alpha$ assume first that $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) = 1$. The first part of the lemma then implies that we have $W^Q_{\bar{t}}(G,\bar{t}) = W^Q_{\bar{t}}(B,\bar{t})$ and $r^Q(G,\bar{t}) + r^Q(B,\bar{t}) = 1$. We can then explicitly solve this system of equations for $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$ as functions of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$. The resulting expressions are

$$
\begin{aligned}
r^Q(G,\bar{t}) &= \frac{M_G\left(r^E_\varnothing(G,\bar{t})\right)^{\frac{1}{\alpha}}}{M_G\left(r^E_\varnothing(G,\bar{t})\right)^{\frac{1}{\alpha}} + M_B\left(r^E_\varnothing(B,\bar{t})\right)^{\frac{1}{\alpha}}}, \\
r^Q(B,\bar{t}) &= \frac{M_B\left(r^E_\varnothing(B,\bar{t})\right)^{\frac{1}{\alpha}}}{M_G\left(r^E_\varnothing(G,\bar{t})\right)^{\frac{1}{\alpha}} + M_B\left(r^E_\varnothing(B,\bar{t})\right)^{\frac{1}{\alpha}}},
\end{aligned}
\tag{49}
$$

where

$$M_G(x) := (T-\bar{t}) \cdot (\tilde{p}_0 z_{\bar{t}} + (1-z_{\bar{t}})\,x)^\alpha + \theta \cdot p_0\,(\pi z_{\bar{t}} + (1-z_{\bar{t}})\,x)^\alpha + \theta \cdot (1-p_0)\,((1-\pi)\,z_{\bar{t}} + (1-z_{\bar{t}})\,x)^\alpha\,,$$

$$M_B(x) := (T-\bar{t}) \cdot ((1-\tilde{p}_0)\,z_{\bar{t}} + (1-z_{\bar{t}})\,x)^\alpha + \theta \cdot p_0\,((1-\pi)\,z_{\bar{t}} + (1-z_{\bar{t}})\,x)^\alpha + \theta \cdot (1-p_0)\,(\pi z_{\bar{t}} + (1-z_{\bar{t}})\,x)^\alpha\,.$$

In case $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) < 1$, the second part of the lemma prescribes that $W^Q_{\bar{t}}(G,\bar{t}) = W^Q_{\bar{t}}(B,\bar{t}) = W^Q_{\bar{t}}(\varnothing)$. Analogously to the previous case, we can solve for $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$ as functions of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ and obtain

$$
\begin{aligned}
r^Q(G,\bar{t}) &= \frac{M_G\left(r^E_\varnothing(G,\bar{t})\right)^{\frac{1}{\alpha}}}{M_G\left(r^E_\varnothing(G,\bar{t})\right)^{\frac{1}{\alpha}} + M_B\left(r^E_\varnothing(B,\bar{t})\right)^{\frac{1}{\alpha}} + (1-z_{\bar{t}}) \cdot (T-\bar{t}+\theta)^{\frac{1}{\alpha}} \cdot \left(1 - r^E_\varnothing(G,\bar{t}) - r^E_\varnothing(B,\bar{t})\right)}, \\
r^Q(B,\bar{t}) &= \frac{M_B\left(r^E_\varnothing(B,\bar{t})\right)^{\frac{1}{\alpha}}}{M_G\left(r^E_\varnothing(G,\bar{t})\right)^{\frac{1}{\alpha}} + M_B\left(r^E_\varnothing(B,\bar{t})\right)^{\frac{1}{\alpha}} + (1-z_{\bar{t}}) \cdot (T-\bar{t}+\theta)^{\frac{1}{\alpha}} \cdot \left(1 - r^E_\varnothing(G,\bar{t}) - r^E_\varnothing(B,\bar{t})\right)},
\end{aligned}
\tag{50}
$$

Note that expressions in (49) can be obtained from the respective ones in (50) substituting $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) = 1$. Therefore, without loss we can restrict ourselves to the case $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) < 1$ and only consider strategy profile given by (50). All above proves existence of the solution for $t = \bar{t}$.

We establish existence for all $t \in \mathcal{S} \backslash \{\bar{t}\}$ proceeding by backward induction. Consider system

$$W^Q_{t_{|\mathcal{S}|-1}}(G, t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(B, t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(G, \bar{t})$$

We next show that this system of equations always has a solution. Consider the following auxiliary system. For any given $c > 0$ assume that $r^Q(G, t_{|\mathcal{S}|-1}) + r^Q(B, t_{|\mathcal{S}|-1}) = c$ and consider equation

$$W^Q_{t_{|\mathcal{S}|-1}}(G, t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(B, t_{|\mathcal{S}|-1}).$$

Then if $r^Q(G, t_{|\mathcal{S}|-1})$ approaches zero, the LHS approaches $+\infty$ while the RHS is constant. Similarly, the RHS strictly dominates the LHS when $r^Q(G, t_{|\mathcal{S}|-1}) = c$. Moreover the LHS is strictly decreasing in $r^Q(G, t_{|\mathcal{S}|-1})$, while the RHS is strictly increasing in it. Therefore by the Intermediate Value Theorem for a given $c > 0$ there exists a unique pair $r^Q(G, t_{|\mathcal{S}|-1}), r^Q(B, t_{|\mathcal{S}|-1})$ such that $r^Q(G, t_{|\mathcal{S}|-1}) + r^Q(B, t_{|\mathcal{S}|-1}) = c$ and $W^Q_{t_{|\mathcal{S}|-1}}(G, t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(B, t_{|\mathcal{S}|-1})$. Also note that both $r^Q(G, t_{|\mathcal{S}|-1})$ and $r^Q(B, t_{|\mathcal{S}|-1})$ are strictly increasing in c. Further for the same $c > 0$ still assume that $r^Q(G, t_{|\mathcal{S}|-1}) + r^Q(B, t_{|\mathcal{S}|-1}) = c$ and consider equality

$$W^Q_{t_{|\mathcal{S}|-1}}(G, t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(G, \bar{t})$$

as an equation in c. The RHS of it is a strictly increasing function of c which approaches $+\infty$ when c approaches 1. As established before, the LHS of it is a strictly decreasing function of c (because $r^Q(G, t_{|\mathcal{S}|-1})$ is strictly increasing in c), which approaches $+\infty$ when c approaches zero. Therefore there exist unique $r^Q(G, t_{|\mathcal{S}|-1})$ and $r^Q(B, t_{|\mathcal{S}|-1})$ such that $W^Q_{t_{|\mathcal{S}|-1}}(G, t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(B, t_{|\mathcal{S}|-1}) = W^Q_{t_{|\mathcal{S}|-1}}(G, \bar{t})$, which finishes the proof. □

The bottom line of the lemma above is that for a given tuple $\left[\mathcal{S}, r^E_\varnothing(G, \bar{t}), r^E_\varnothing(B, \bar{t})\right]$, a strategy profile that constitutes a relay equilibrium is a unique solution to a particular system of algebraic equations. Moreover, solution to this system always exists if $w(\beta) = \beta^\alpha$ and $w^c(\beta) = \theta \cdot \beta^\alpha$. Representing a strategy profile as a solution to a system of equations allows us to compare equilibrium strategies and, therefore, report informativeness across different relay equilibria employing the arguments similar to the Implicit Function Theorem.

In all further lemmas it is assumed that strategy profile $r^\gamma_\eta(m, t)$ and all associated equilibrium objects such as values $W^Q_t(m, \tau)$, belief profiles b, p, and informativeness measures $i(m, t)$ constitute a solution to either system (47) or system (48) for a given tuple $\left[\mathcal{S}, r^E_\varnothing(G, \bar{t}), r^E_\varnothing(B, \bar{t})\right]$, and therefore are understood as functions of $\left[\mathcal{S}, r^E_\varnothing(G, \bar{t}), r^E_\varnothing(B, \bar{t})\right]$.

The next lemma establishes that whenever $w(\beta) = \beta^\alpha$ and $w^c(\beta) = \theta \cdot \beta^\alpha$, it is true that strategies that constitute a solution to either system (47) or system (48) are continuously differentiable in $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ at $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})=1$. The same will then be true of all associated equilibrium objects $W^Q_t(m,\tau)$, b and $i(m,t)$, as they all are continuously differentiable functions of the strategies. The statement of this lemma is valid for any continuously differentiable $w(\cdot)$, but the statement for this particular functional form is enough for the needs of the paper and is significantly easier to prove. Lemma 7 allows us to further omit the consideration of the case $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})=1$ and without loss assume in further propositions that $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})<1$.

Lemma 7. *Suppose $w(\beta)=\beta^\alpha$ and $w^c(\beta)=\theta\cdot\beta^\alpha$, and strategy profile $\{r^\gamma_\eta(m,t)\}$ solves either system* (47) *or system* (48). *Then $r^\gamma_\eta(m,t)$ is a continuously differentiable function of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ for all $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})\leqslant 1$.*

Proof. First note that $r^\gamma_\eta(m,t)$ exists by Lemma 6. Next, the strategy profile for the expert is fixed by the premise of Lemma 6 and is therefore a continuously differentiable function of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$. Therefore we are left to establish that $r^Q(m,t)$ is a continuously differentiable function of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ for all $m\in\{G,B\}$ and all $t\in\mathcal{S}$. As expressions in (49) coincide with the ones in (50) for $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})=1$, without loss, we restrict ourselves to the case $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})<1$. Both expressions in (50) are continuously differentiable functions of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ for $r^E_\varnothing(G,\bar{t})+r^E_\varnothing(B,\bar{t})\leqslant 1$. Thus, it is left to show the same for $r^Q(G,t)$ and $r^Q(B,t)$ for $t\in\mathcal{S}\backslash\{\bar{t}\}$. We proceed using backward induction. Consider two equalities

$$W^Q_{t_{|\mathcal{S}|-1}}\left(G,t_{|\mathcal{S}|-1}\right)=W^Q_{t_{|\mathcal{S}|-1}}\left(G,\bar{t}\right),$$
$$W^Q_{t_{|\mathcal{S}|-1}}\left(B,t_{|\mathcal{S}|-1}\right)=W^Q_{t_{|\mathcal{S}|-1}}\left(B,\bar{t}\right).$$

Given $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$, they constitute a system of equations on $r^Q\left(G,t_{|\mathcal{S}|-1}\right)$ and $r^Q\left(B,t_{|\mathcal{S}|-1}\right)$. Moreover, because $w(\beta)=\beta^\alpha$, both $r^Q\left(G,t_{|\mathcal{S}|-1}\right)$ and $r^Q\left(B,t_{|\mathcal{S}|-1}\right)$ do not depend on $b_{t_{|\mathcal{S}|-2}}$ – that is, on strategies $r^Q(G,t)$ and $r^Q(B,t)$ for $t\leqslant t_{|\mathcal{S}|-2}$ – but only on $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$. Therefore, by the Implicit Function Theorem, $r^Q\left(G,t_{|\mathcal{S}|-1}\right)$ and $r^Q\left(B,t_{|\mathcal{S}|-1}\right)$ are continuously differentiable functions of $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$, which eventually implies that they are continuously differentiable functions of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$. Proceeding backwards we establish the claim for all $r^Q(G,t)$ and $r^Q(B,t)$ for $t\in\mathcal{S}$. $\square$

The two following lemmas are mostly technical and provide little intuition for the main problem.

Lemma 8. *Suppose*

$$m(x) = \left(\chi_1 (a_1 + bx)^\alpha + \ldots + \chi_k (a_k + bx)^\alpha\right)^{\frac{1}{\alpha}} - bx$$

for $k \geqslant 2$*,* $b > 0$*,* $\sum_{i=1}^{k} \chi_i = 1$*,* $a_1, \ldots, a_k \geqslant 0$ *with* $a_i, a_j > 0$ *for some* $i, j \in \{1, \ldots, k\}$*,* $i \neq j$*. Then* $m(x)$ *is strictly decreasing when* $\alpha > 1$ *and is strictly increasing when* $\alpha < 1$*.*

Proof. Begin by observing that

$$\frac{1}{b} \cdot \frac{dm(x)}{dx} = \frac{\chi_1 (a_1 + bx)^{\alpha-1} + \ldots + \chi_k (a_k + bx)^{\alpha-1}}{\left(\chi_1 (a_1 + bx)^\alpha + \ldots + \chi_k (a_k + bx)^\alpha\right)^{\frac{\alpha-1}{\alpha}}} - 1.$$

First, if $\alpha > 1$ then, since x^k is strictly convex for $k > 1$, we have

$$\left(\chi_1 (a_1 + bx)^{\alpha-1} + \ldots + \chi_k (a_k + bx)^{\alpha-1}\right)^{\frac{\alpha}{\alpha-1}} < \chi_1 (a_1 + bx)^\alpha + \ldots + \chi_k (a_k + bx)^\alpha, \tag{51}$$

and therefore $\frac{dm(x)}{dx} < 0$ if $\alpha > 1$.

Second, if $\alpha < 1$ then, because x^k is strictly convex for $k < 0$, we still have (51) satisfied. Therefore, because $\frac{\alpha-1}{\alpha} < 0$, we have

$$\chi_1 (a_1 + bx)^{\alpha-1} + \ldots + \chi_k (a_k + bx)^{\alpha-1} > \left(\chi_1 (a_1 + bx)^\alpha + \ldots + \chi_k (a_k + bx)^\alpha\right)^{\frac{\alpha-1}{\alpha}},$$

and therefore $\frac{dm(x)}{dx} > 0$ if $\alpha < 1$. □

Lemma 9. *Suppose* $w(\beta) = \beta^\alpha$*, and strategy profile* $\{r^\gamma_\eta(m,t)\}$ *solves either system* (47) *or system* (48)*. Then if* $\alpha < 1$ *for any* $m \in \{G, B\}$ *we have*

$$\frac{\partial}{\partial r^E_\varnothing(m,\bar{t})} \left(\frac{(1 - z_{\bar{t}}) \cdot \left(1 - r^E_\varnothing(G,\bar{t}) - r^E_\varnothing(B,\bar{t})\right)}{1 - r^Q(G,\bar{t}) - r^Q(B,\bar{t})} \right) > 0,$$

while if $\alpha > 1$ *for any* $m \in \{G, B\}$ *we have*

$$\frac{\partial}{\partial r^E_\varnothing(m,\bar{t})} \left(\frac{(1 - z_{\bar{t}}) \cdot \left(1 - r^E_\varnothing(G,\bar{t}) - r^E_\varnothing(B,\bar{t})\right)}{1 - r^Q(G,\bar{t}) - r^Q(B,\bar{t})} \right) < 0.$$

Additionally, for any $m \in \{G, B\}$ *and any* $t \in \mathcal{S}$ *we have that* $W^Q_{t_1}(m,t)$ *is strictly increasing in* $r^E_\varnothing(m,\bar{t})$ *if* $\alpha < 1$*, and* $W^Q_{t_1}(m,t)$ *is strictly decreasing in* $r^E_\varnothing(m,\bar{t})$ *if* $\alpha < 1$*.*

Proof. Lemmas 6 and 7 imply that $r^Q(m,t)$ exists for all $m \in \{G, B\}$ and $t \in \mathcal{S}$ and is continuously differentiable in $r^E_\varnothing(G,\bar t)$ and $r^E_\varnothing(B,\bar t)$. Next, from (50) we can calculate that

$$\frac{(1-z_{\bar t})\cdot\left(1-r^E_\varnothing(G,\bar t)-r^E_\varnothing(G,\bar t)\right)}{1-r^Q(G,\bar t)-r^Q(B,\bar t)}=\frac{M_G\left(r^E_\varnothing(G,\bar t)\right)^{\frac{1}{\alpha}}+M_B\left(r^E_\varnothing(B,\bar t)\right)^{\frac{1}{\alpha}}}{(T-\bar t+\theta)^{\frac{1}{\alpha}}}+(1-z_{\bar t})\cdot\Big(1-r^E_\varnothing(G,\bar t)-r^E_\varnothing(B,\bar t)\Big),$$

which is the sum of two functions of the form from Lemma 8 and a constant. Therefore, the first statement of the Lemma follows directly from Lemma 8.

Next we establish the second claim. By Lemma 7 we can assume without loss that $r^E_\varnothing(G,\bar t)+r^E_\varnothing(B,\bar t)<1$. What follows is the proof for the case $\alpha<1$ (case $\alpha>1$ is analogous). For a given $m\in\{G,B\}$ fix some $r^E_\varnothing(m,\bar t)<\tilde r^E_\varnothing(m,\bar t)$ and $r^E_\varnothing(-m,\bar t)=\tilde r^E_\varnothing(-m,\bar t)$. Also denote the respective strategy profiles that solve system (48) for $\left[\mathcal{S},r^E_\varnothing(G,\bar t),r^E_\varnothing(B,\bar t)\right]$ and $\left[\mathcal{S},\tilde r^E_\varnothing(G,\bar t),\tilde r^E_\varnothing(B,\bar t)\right]$ as $\{r^\gamma_\eta(m,t)\}$ and $\{\tilde r^\gamma_\eta(m,t)\}$. Denote by b and $\tilde b$ the respective beliefs, and by $W^\gamma_t(m,\tau)$ and $\tilde W^\gamma_t(m,\tau)$ the respective values from reports.

Assume that $W^Q_{t_1}(m,t)\geqslant\tilde W^Q_{t_1}(m,t)$. Then $r^Q(m,t_1)\leqslant\tilde r^Q(m,t_1)$ for $m\in\{G,B\}$. From (44) we then get that $b_{t_1}\leqslant\tilde b_{t_1}$. This, in turn, implies that $r^Q(G,t_2)\leqslant\tilde r^Q(G,t_2)$ and $r^Q(B,t_2)\leqslant\tilde r^Q(B,t_2)$ because $W^Q_{t_1}(m,t_1)=W^Q_{t_1}(m,t_2)$ and $\tilde W^Q_{t_1}(m,t_1)=\tilde W^Q_{t_1}(m,t_2)$ for $m\in\{G,B\}$. Iterating this logic further, we get $b_{t_{|\mathcal{S}-1|}}\leqslant\tilde b_{t_{|\mathcal{S}-1|}}$. From the first part of the lemma we know that

$$\frac{(1-z_{\bar t})\cdot\left(1-\tilde r^E_\varnothing(G,\bar t)-\tilde r^E_\varnothing(G,\bar t)\right)}{1-\tilde r^Q(G,\bar t)-\tilde r^Q(B,\bar t)}>\frac{(1-z_{\bar t})\cdot\left(1-r^E_\varnothing(G,\bar t)-r^E_\varnothing(G,\bar t)\right)}{1-r^Q(G,\bar t)-r^Q(B,\bar t)},$$

and therefore $W^Q_{t_1}(\varnothing)<\tilde W^Q_{t_1}(\varnothing)$. This gives us a contradiction with the initial assumption $W^Q_{t_1}(m,t)\geqslant\tilde W^Q_{t_1}(m,t)$ because we must have $W^Q_{t_1}(\varnothing)=W^Q_{t_1}(G,t_1)$ and $\tilde W^Q_{t_1}(\varnothing)=\tilde W^Q_{t_1}(G,t_1)$. □

Proof of Proposition 4. We first show that for any set of parameters $|\mathcal{S}|=1$, $r^E_\varnothing(G,\bar t)>0$, and $r^E_\varnothing(B,\bar t)>0$, the informative equilibrium with given parameters exists. Proposition 2 and the values of $r^E_\varnothing(m,t)$ pin down the expert's strategy. We next show that there exists such a quack's strategy $r^Q(m,\bar t)$ that conditions in Lemma 6 are satisfied, which proves this part of the proposition. Note also that for singleton $\mathcal{S}$ we have $z_{\bar t}=F(t)$.

The first condition one needs to check in order to establish existence is $W^Q_{\bar t}(G,\bar t)=W^Q_{\bar t}(B,\bar t)$, which can be written as

$$\begin{aligned}(T-\bar t)\cdot w\left(\beta(G,\bar t)\right)+p_0w^c\left(\beta^G(G,\bar t)\right)+(1-p_0)\,w^c\left(\beta^B(G,\bar t)\right)=\\=(T-\bar t)\cdot w\left(\beta(B,\bar t)\right)+p_0w^c\left(\beta^G(B,\bar t)\right)+(1-p_0)\,w^c\left(\beta^B(B,\bar t)\right).\end{aligned}\tag{52}$$

From (42) and (43) we see that the LHS is strictly decreasing in $r^Q(G,\bar t)$, and the RHS is strictly increasing in $r^Q(G,\bar t)$. Moreover, all six terms in (52) are always positive irrespectively of $r^Q(G,\bar t)$ because $r^E_\varnothing(G,\bar t)>0$ and $r^E_\varnothing(B,\bar t)>0$. Therefore when $r^Q(G,\bar t)=0$ the LHS strictly dominates the RHS, and when $r^Q(G,\bar{t}) = 1$ the RHS strictly dominates the LHS. By the Intermediate Value Theorem, there exists unique $r^Q(G,\bar{t})$ (and therefore $r^Q(B,\bar{t})$) such that (52) is satisfied.

If $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) = 1$ then we are done. Strictly speaking, this is enough to prove the statement. However, for sake of completeness we show that the equilibrium exists in case $r^E_\varnothing(G,\bar{t}) + r^E_\varnothing(B,\bar{t}) < 1$ as well. To do this we need to ensure that $W^Q_{\bar{t}}(G,\bar{t}) = W^Q_{\bar{t}}(\varnothing)$, i.e., that the value of not making a report at $\bar{t}$ is equal to the value of making a report:

$$(T-\bar{t}) \cdot w\left(\beta(G,\bar{t})\right) + p_0 w^c\left(\beta^G(G,\bar{t})\right) + (1-p_0)\, w^c\left(\beta^B(G,\bar{t})\right) = (T-\bar{t}) \cdot w\left(\beta_{\bar{t}}\right) + w^c\left(\beta_{\bar{t}}\right). \quad (53)$$

By the same logic as above, we know that for any given $c > 0$ there exist unique $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$ such that (52) is satisfied and $r^Q(G,\bar{t}) + r^Q(B,\bar{t}) = c$. Further, note that $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ that are obtained as a solution to this auxiliary system are both increasing in c.[90] Finally, consider (53) as an equation in c. From the previous observation it follows that its LHS is decreasing in c, while the RHS is increasing in c. If $c = 0$ then the LHS dominates the RHS, and if $c = 1$ the RHS dominates the LHS. Therefore, by the Intermediate Value Theorem there exists unique c such that (53) is satisfied. Solving (52) using this c gives a pair $r^Q(G,\bar{t})$ and $r^Q(B,\bar{t})$ that uniquely solves the original system of (52) and (53).

To prove the second part of the proposition we construct a relay reticent equilibrium for a given $\mathcal{S}$ with $|\mathcal{S}| \geqslant 2$ and $r^E_\varnothing(G,\bar{t}) = r^E_\varnothing(B,\bar{t}) = 0$. Since states are symmetric, for any $t \in \mathcal{S}$ the quack is indifferent between reports (G,t) and (B,t) if and only if $r^Q(G,t) = r^Q(B,t)$. Thus, we are only left to ensure the indifference between a report and no report for the quack and to verify that the constructed equilibrium is incentive compatible for informed expert. Define $g := g(G,t_1) = g(B,t_1)$. Then $W^Q_{t_1}(G,t_1) = W^Q_{t_1}(B,t_1)$ is equal to

$$(T-t_1) \cdot w\left(\beta_0 \cdot e^g\right) + p_0 w^c\left(\beta_0 \cdot \frac{\pi}{\tilde{p}_0} e^g\right) + (1-p_0)\, w^c\left(\beta_0 \cdot \frac{(1-\pi)}{\tilde{p}_0} e^g\right).$$

From the expression above we see that the value of g fully determines the value that the quack gets in equilibrium. In a relay equilibrium the expert's strategy is fixed, so smaller g means larger $r^Q(G,t_1)$ and $r^Q(B,t_1)$. Larger $r^Q(G,t_1)$ and $r^Q(B,t_1)$, in turn, imply higher b_{t_1}. Finally, because the quack must be indifferent between reports at t_1 and t_2, higher b_{t_1} implies larger $r^Q(G,t_2)$ and $r^Q(B,t_2)$. All in all, it means that the payoff that the quack receives by not making a report is

[90] At least one of $r^E_\varnothing(G,\bar{t})$ and $r^E_\varnothing(B,\bar{t})$ must be higher for a higher c, and (52) implies that higher $r^E_\varnothing(G,\bar{t})$ implies higher $r^E_\varnothing(B,\bar{t})$.

point-wise strictly decreasing in g. When $g = 0$ we have that $b_t = b_0$ for all $t \in \mathcal{T}$ (remember that $r^E_\varnothing(G, \bar{t}) = r^E_\varnothing(B, \bar{t}) = 0$, so following the logic from Proposition 9 we have $g(m,t) = 0$ for all $t \in \mathcal{T}$), therefore value of making no report by the end of period T evaluated at t_1 is equal to $(T - t_1) \cdot w(\beta_0) + w^c(\beta_0)$. When $g \to -\infty$ we have that the value of making no report strictly dominates the value of making a report. When $g = 0$ we have

$$(T - t_1) \cdot w(\beta_0) + p_0 w^c\left(\beta_0 \cdot \frac{\pi}{\tilde{p}_0}\right) + (1 - p_0)\, w^c\left(\beta_0 \cdot \frac{(1-\pi)}{\tilde{p}_0}\right) \geqslant (T - t_1) \cdot w(\beta_0) + w^c(\beta_0)$$

because $w^c(\cdot)$ is convex. Therefore, by the Intermediate Value Theorem, there exists a unique $g \leqslant 0$ such that the quack is indifferent between making a report and not making a report. Finally, because $g(G, t_1) = g(B, t_1) = g \leqslant 0$ we have $b(m, t_1) \leqslant b_0 \leqslant b_{t_1}$. From Proposition 9 for convex $w^c(\cdot)$ we know that it implies that $b^m(m,t)$ – and, consequently, $\mathit{\Delta} w_\eta(m,t)$ for $m = \eta$ – are decreasing on $\mathcal{S}$, which verifies that $r^E_\eta(\eta, t) = 1$ is an optimal strategy for the expert, i.e., the constructed profile indeed constitutes an equilibrium.

To prove the third part assume the contrary: there exists $\mathcal{S}$ with $|\mathcal{S}| \geqslant 3$ such that informative equilibrium with respective strategy profile $\{r^\gamma_\eta(m,t)\}$ for $t \in \mathcal{S}$ exists. By Lemma 5 we can assume without loss that the equilibrium is a relay one. By Lemma 6 we know that there exists a strategy profile $\{\tilde{r}^\gamma_\eta(m,t)\}$ (and associated belief profile $\tilde{b}$ and value function $\tilde{W}^\gamma_t(m,\tau)$) for the same $\mathcal{S}$ with $\tilde{r}^E_\varnothing(G,t) = \tilde{r}^E_\varnothing(B,t) = 0$ which solves system (48). We next show that this profile constitutes a relay equilibrium on $\mathcal{S}$. The only condition that needs to be verified is that this profile is incentive compatible for informed expert. By the proof of Proposition 9, for $\mathcal{S}$ with $|\mathcal{S}| \geqslant 3$ this is equivalent to verifying that $\tilde{b}_{t_1} \geqslant \tilde{b}(m, t_1)$ for $m \in \{G, B\}$ because $\tilde{r}^E_\varnothing(G,t) = \tilde{r}^E_\varnothing(B,t) = 0$.

In the original equilibrium we have $b_{t_1} \geqslant b(m, t_1)$ by the same Proposition 9. By Lemma 9, because $r^E_\varnothing(m,t) \geqslant \tilde{r}^E_\varnothing(m,t)$ for $m \in \{G, B\}$ and $\alpha < 1$, we have $W^Q_{t_1}(m, t_1) > \tilde{W}^Q_{t_1}(m, t_1)$. This implies that $r^Q(m, t_1) < \tilde{r}^Q(m, t_1)$ for $m \in \{G, B\}$, and therefore $\tilde{b}_{t_1} > b_{t_1} \geqslant b(m, t_1) > \tilde{b}(m, t_1)$ for $m \in \{G, B\}$, which completes the argument.

We have established the existence of the relay equilibrium on $\mathcal{S}$ with $\tilde{r}^E_\varnothing(G,t) = \tilde{r}^E_\varnothing(B,t) = 0$. By Proposition 9 there exists $m \in \{G, B\}$ such that $\tilde{b}(m, t_1) \leqslant b_0$, and therefore

$$\tilde{W}^Q_{t_1}(m, t_1) \leqslant (T - t_1) \cdot w(\beta_0) + p_0 w^c\left(\beta_0 \cdot \frac{\pi}{\tilde{p}_0}\right) + (1 - p_0)\, w^c\left(\beta_0 \cdot \frac{1-\pi}{\tilde{p}_0}\right).$$

At the same time, because in such equilibrium $b_t \geqslant b_0$ for all $t \in \mathcal{S}$ (again by Proposition 9), we have

$$\tilde{W}^Q_{t_1}(\varnothing) \geqslant (T - t_1) \cdot w(\beta_0) + w^c(\beta_0).$$

Finally, $\tilde{W}^Q_{t_1}(m, t_1) = \tilde{W}^Q_{t_1}(\varnothing)$ implies

$$p_0 w^c\left(\beta_0 \cdot \frac{\pi}{\tilde{p}_0}\right) + (1 - p_0)\, w^c\left(\beta_0 \cdot \frac{1-\pi}{\tilde{p}_0}\right) \geqslant w^c(\beta_0). \tag{54}$$

If $w^c(\cdot)$ is strictly concave then (54) can not be satisfied, which gives us the contradiction. $\square$

Proof of Proposition 5. Denote by $W^\gamma_t(m,\tau)$ and $\tilde{W}^\gamma_t(m,\tau)$ the respective values of making report and by b and $\tilde{b}$ the beliefs for strategy profiles $\{r^\gamma_\eta(m,t)\}$ and $\{\tilde{r}^\gamma_\eta(m,t)\}$.

To prove the first part of the proposition, we first show that $W^Q_{t_1}(m,t_1) \leqslant \tilde{W}^Q_{t_1}(m,t_1)$ for $m \in \{G,B\}$. Assume the contrary, i.e., that $W^Q_{t_1}(m,t_1) > \tilde{W}^Q_{t_1}(m,t_1)$ for $m \in \{G,B\}$.[91] Then it directly implies $r^Q(m,t_1) < \tilde{r}^Q(m,t_1)$ for $m \in \{G,B\}$. From (44) it then follows that $b_{t_1} < \tilde{b}_{t_1}$. This, in turn, implies that $r^Q(G,t_2) < \tilde{r}^Q(G,t_2)$ and $r^Q(B,t_2) < \tilde{r}^Q(B,t_2)$ because $W^Q_{t_1}(m,t_1) = W^Q_{t_1}(m,t_2)$ and $\tilde{W}^Q_{t_1}(m,t_1) = \tilde{W}^Q_{t_1}(m,t_2)$ for $m \in \{G,B\}$. Iterating this logic further, we obtain that $b_t < \tilde{b}_t$ for all $t \in \mathcal{S}$. Additionally, by Proposition 9 we have $b_{t_k} \leqslant b_{t_{k+1}} \leqslant \ldots \leqslant b_{t_{k+n}}$ (we can extend the argument to $b_{t_{k+n}}$ because $\tilde{r}^E_\varnothing(m,t_{n+k}) = 0$ for $m \in \{G,B\}$). Therefore, $W^Q_{t_1}(\varnothing) < \tilde{W}^Q_{t_1}(\varnothing)$. Making no report is an on-path action in both equilibria, thus $W^Q_{t_1}(G,t_1) = W^Q_{t_1}(\varnothing)$ and $\tilde{W}^Q_{t_1}(G,t_1) = \tilde{W}^Q_{t_1}(\varnothing)$. Consequently, $W^Q_{t_1}(m,t_1) < \tilde{W}^Q_{t_1}(m,t_1)$, which gives us a contradiction with the initial assumption.

Condition $W^Q_{t_1}(m,t) \leqslant \tilde{W}^Q_{t_1}(m,t)$ directly implies that $r^Q(m,t_1) \geqslant \tilde{r}^Q(m,t_1)$ – since in a relay equilibrium the strategy of the expert is fixed, – and therefore $|i(m,t)| \leqslant |\tilde{i}(m,t)|$ for all $t \in \mathcal{S}$. Finally, $|i(m,t)| = 0$ for any $t \notin \mathcal{S}$, meaning that $|i(m,t)| < |\tilde{i}(m,t)|$ for $t \in \tilde{\mathcal{S}} \backslash \mathcal{S}$. $\square$

Proof of Proposition 8. In an ideal equilibrium, $r^E_\eta(\eta,t) = 1$ and $r^E_\varnothing(m,t) = r^Q(m,t) = 0$ for $\eta, m \in \{G,B\}$ and all $t \in \mathcal{S}$. Report (η,t) at t yields maximal continuation reputation to the expert with information $\eta \in \{G,B\}$, so is trivially optimal at the time when he receives his private signal. The uninformed expert's preference for staying silent at any t is at least as high as that of the quack (due to the option value of receiving news in the future and obtaining the maximal continuation payoff). Therefore, it is enough to verify that the quack prefers to stay silent at

[91] Because $W^Q_{t_1}(G,t_1) = W^Q_{t_1}(B,t_1)$ and $\tilde{W}^Q_{t_1}(G,t_1) = \tilde{W}^Q_{t_1}(B,t_1)$, we have that either $W^Q_{t_1}(m,t_1) > \tilde{W}^Q_{t_1}(m,t_1)$ for both $m \in \{G,B\}$ or $W^Q_{t_1}(m,t_1) < \tilde{W}^Q_{t_1}(m,t_1)$ for both $m \in \{G,B\}$.

every point of the support. Since after any report the reputation jumps to $\bar{w} := \lim_{x \to +\infty} w(x)$ and to $\bar{w}^c := \lim_{x \to +\infty} w^c(x)$ after state is revealed, in case of no report it must decrease, as b_t is a martingale. Therefore, $W_{t_1}^Q(m,t)$ is maximized at $t = t_1$ (and any m). Report (G, t_1) and report (B, t_1) yield, respectively,

$$W_{t_1}^Q(G,t_1) = (T - t_1) \cdot \bar{w} + p_0 \cdot \bar{w}^c,$$

$$W_{t_1}^Q(B,t_1) = (T - t_1) \cdot \bar{w} + (1 - p_0) \cdot \bar{w}^c.$$

We have assumed $p_0 \geqslant \frac{1}{2}$, and therefore $W_{t_1}^Q(G,t_1) \geqslant W_{t_1}^Q(B,t_1)$. Using (4), (38), and (39), one can calculate the flow payoff from staying silent until t, which equals $w(\beta_0(1 - F(t)))$. Therefore, value from not making a report until the last point of $\mathcal{S}$, as evaluated at t_1, equals

$$W_{t_1}^Q(\varnothing) = \sum_{k=1}^{k=|\mathcal{S}|} (t_{k+1} - t_k) \cdot w(\beta_0(1 - F(t_k))) + (T - \bar{t}) \cdot w(\beta_0(1 - F(\bar{t}))) + w^c(\beta_0(1 - F(\bar{t}))).$$

Staying silent is optimal if and only if $W_{t_1}^Q(G,t_1) \leqslant W_{t_1}^Q(\varnothing)$. This requires that both $\bar{w}$ and $\bar{w}^c$ to be finite, while if $w(\cdot)$ or $w^c(\cdot)$ is strictly increasing and convex, it must be that $\bar{w} = +\infty$. $\square$

Appendix for "Bad News Turned Good: Reversal under Censorship"

All statements below fix some strategy profile $(r^L(p), r^H(p))_{p\in[0,1]^2}$, which in turn produces functions $D(p)$ and $f^s(p)$. Some statements further require this strategy profile to constitute an equilibrium.

Lemmas 1 and 2 are used heavily throughout the Appendix. They are monolithic in essence, but it proved more convenient to stagger their proofs for different bands, since they use different supplementary results.

Lemma 10. *1. $D(p) \in [-1, -q)$ if and only if $f^s(p) > p^s$.*

2. $D(p) = -q$ if and only if either $f^s(p) = p^s$ or $r^H(p) = r^L(p) = 0$.

3. $D(p) \in (-q, 1-q]$ if and only if $f^s(p) < p^s$.

Proof. To show the first claim, observe that $D(p) < -q$ is equivalent to

$$(1-q)\cdot(1-r^H(p)) - (1-r^L(p)) < -q$$
$$\Leftrightarrow (1-q)\cdot r^H(p) - r^L(p) > 0$$
$$\Leftrightarrow \left(\frac{f^s(p)}{1-f^s(p)}\right)\cdot\left(\frac{p^s}{1-p^s}\right)^{-1} \equiv (1-q)\cdot\frac{r^H(p)}{r^L(p)} > 1.$$

Two other claims can be obtained by reversing the inequalities or equating both sides. Finally, if $r^H(p) = r^L(p) = 0$, then (14) directly gives that $D(p) = (1-q) - 1 = -q$. □

Lemma 11. *For any $k \geqslant 0$ the following hold:*

1. *For all $p = (p^n, p^s) \in \mathcal{B}_k^{\uparrow}$, if there exists $\tilde{p} = (p^n, \tilde{p}^s)$ with $\tilde{p}^s \in [\bar{p}, p^s]$ and $D(\tilde{p}) \geqslant 0$, then $\tau(p) = +\infty$. Otherwise $\tau(p)$ can be represented as*

$$\tau(p) = -\int_{\bar{p}}^{p^s} \frac{1}{\lambda z(1-z)\cdot\pi(p^n, z)D(p^n, z)}dz. \tag{55}$$

2. *For any $p = (p^n, p^s) \in \mathcal{B}_k^{\uparrow}$, if $D(p^n, \tilde{p}^s) < 0$ for all $\tilde{p}^s \in [\bar{p}, p^s]$ and $\tau(p) < +\infty$, then $\tau(p^n, \cdot)$ is differentiable in its second argument at p^s.*[92]

[92] At p with $p^s = \bar{p}$ by the derivative of $\tau(p^n, \cdot)$ we understand its right derivative.

3. If $D(p) \leqslant -\varepsilon < 0$ for all $p \in \mathcal{B}_k^{\uparrow}$, then $\tau(p)$ is finite for all $p \in \mathcal{B}_k^{\uparrow}$.

4. Suppose $g(p) : \mathcal{B}_k^{\uparrow} \to [\bar{p}, 1]$ is defined indirectly as $\tau(f^n(p^n), g(p)) = \psi(\tau(p))$ for some differentiable and strictly increasing function ψ, and $\tau(p)$ is finite for any $p \in \mathcal{B}_k^{\uparrow} \cup \mathcal{B}_{k-1}^{\uparrow}$ with $p^s < 1$, strictly increasing and differentiable in p^s on $[\bar{p}, 1)$.[93] Then $g(p)$ is a strictly increasing and differentiable function of p^s. In particular, we have the following representation:

$$\ln\left(\frac{g(p)}{1-g(p)}\right) = J(p) + \ln\left(\frac{p^s}{1-p^s}\right)$$

where $J(p)$ is a differentiable function of p^s.

Proof. 1. If there exists $\tilde{p} = (p^n, \tilde{p}^s)$ with $\tilde{p}^s \in [\bar{p}, p^s]$ and $D(\tilde{p}) \geqslant 0$, then p_t never reaches $\mathcal{B}_k^{\downarrow}$, so $\tau(p) = +\infty$ by definition. Now let p_t^s denote the solution to (14) with the initial condition $p_0^s = p^s$. If $D(p^n, \tilde{p}^s) < 0$ for all $\tilde{p}^s \in [\bar{p}, p^s]$, p_t^s is a strictly decreasing function of t. Therefore, there exists an inverse function $t(p_t^s)$ measuring the time it takes for belief to drift from the initial value p^s to p_t^s. Its derivative is given by

$$\frac{dt(p_t^s)}{dp_t^s} = \left(\lambda p_t^s(1-p_t^s) \cdot \pi(p^n, p_t^s) D(p^n, p_t^s)\right)^{-1},$$

and $t(p^s) = 0$. Therefore, $t(p_t^s) = \int_{p^s}^{p_t^s} \frac{1}{\lambda z(1-z)\cdot\pi(p^n,z)D(p^n,z)}dz$. As $D(\bar{p}) < 0$, the threshold is crossed in zero time. Then substituting $p_t^s = \bar{p}$ we get the result.[94]

2. If $D(p^n, \tilde{p}^s) < 0$ for all $\tilde{p}^s \in [\bar{p}, p^s]$, then representation (77) is valid. Taking the derivative with respect to p^s we get

$$\frac{d\tau(p)}{dp^s} = -\left(\lambda p^s(1-p^s) \cdot \pi(p^n, p^s) D(p^n, p^s)\right)^{-1}. \tag{56}$$

As long as $0 < \bar{p} \leqslant p^s < 1$ and $D(p^n, p^s) < 0$, the derivative is finite and positive.

3. In case $D(p) \leqslant -\varepsilon < 0$ the improper integral in (77) converges for any p and therefore $\tau(p) < +\infty$.

4. Differentiability of $g(p)$ follows directly from the differentiability and monotonicity of a composition and an inverse function. Differentiability of $J(p)$ is then straightforward as

[93] If $p \in \mathcal{B}_0$, then we let $\mathcal{B}_{k-1} = \mathcal{B}_k = \mathcal{B}_0$.

[94] This proof does not imply that the integral converges. Hence even if $D(p^n, \tilde{p}^s) < 0$ for all $\tilde{p}^s \in [\bar{p}, p^s]$, it may still be that $\tau(p) = +\infty$.

$\ln\left(\frac{g(p)}{1-g(p)}\right) - \ln\left(\frac{p^s}{1-p^s}\right)$ is a sum of differentiable functions and is therefore differentiable.

□

Lemma 12. *1. $D(p) \geqslant -q+\varepsilon$ for some $\varepsilon \in (0,q]$ implies $\ln\left(\frac{f^s(p)}{1-f^s(p)}\right) - \ln\left(\frac{p^s}{1-p^s}\right) \leqslant \ln(1-\varepsilon)$.*

2. $D(p) \leqslant -q-\varepsilon$ for some $\varepsilon \in (0,1-q]$ implies $\ln\left(\frac{f^s(p)}{1-f^s(p)}\right) - \ln\left(\frac{p^s}{1-p^s}\right) \geqslant \ln(1+\varepsilon)$.

Proof. We prove only the first claim, the second one is analogous. $D(p) \geqslant -q+\varepsilon$ implies

$$-(1-q)\cdot r^H(p) + r^L(p) \geqslant \varepsilon$$

and further

$$\begin{aligned}\ln\left(\frac{f^s(p)}{1-f^s(p)}\right) - \ln\left(\frac{p^s}{1-p^s}\right) &= \ln\left(\frac{(1-q)\cdot r^H(p)}{r^L(p)}\right)\\ &\leqslant \ln\left(1-\frac{\varepsilon}{r^L(p)}\right) \leqslant \ln(1-\varepsilon).\end{aligned}$$

□

Proof of Lemma 1 for $\mathcal{B}_0^{\uparrow}$. Suppose there exists $\tilde{p} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}_0^{\uparrow}$ with $\tau(\tilde{p}) = +\infty$. Then consider states $p_{inf,1} := (\tilde{p}^n, p^s_{inf,1})$ and $p_{inf,2} := (f^n(\tilde{p}^n), p^s_{inf,2})$, where $p^s_{inf,1} = \inf\{p^s \mid \tau(\tilde{p}^n, p^s) = +\infty\}$ and $p^s_{inf,2} = \inf\{p^s \mid \tau(f^n(\tilde{p}^n), p^s) = +\infty\}$.[95] We start by showing that

$$\ln\left(\frac{p^s_{inf,1}}{1-p^s_{inf,1}}\right) - \ln\left(\frac{p^s_{inf,2}}{1-p^s_{inf,2}}\right) \geqslant -\ln\left(1-\frac{q}{2}\right). \tag{57}$$

By Lemma 11 there can be three (mutually non-exclusive) sub-cases to consider.

Case 1 $D(p_{inf,1}) \geqslant 0$. Then $\tau(p_{inf,1}) = +\infty$, and $r^L(p_{inf,1}) > 0$.[96] Therefore, a low-type seller must weakly prefer to disclose a bad review, and thus $\tau(f(p_{inf,1})) = +\infty$. Then $p^s_{inf,2} \leqslant f^s(p_{inf,1})$ by definition of $p^s_{inf,2}$, and $\ln\left(\frac{f^s(p_{inf,1})}{1-f^s(p_{inf,1})}\right) - \ln\left(\frac{p^s_{inf,1}}{1-p^s_{inf,1}}\right) \leqslant \ln(1-q)$ by Lemma 12, which together imply (57).

Case 2 $D(p_{inf,1}) < 0$ and there exists a sequence $\{\tilde{p}^s_k\}$ such that $\tilde{p}^s_k \downarrow p^s_{inf,1}$ and $D(\tilde{p}_k) > -\frac{1}{k}$, where $\tilde{p}_k := (\tilde{p}^n, \tilde{p}^s_k)$. Then for any $\varepsilon > 0$ and sufficiently high K we have $D(\tilde{p}_K) > -\frac{q}{4}$,

[95] As the set is non-empty and bounded from below by $\bar{p}$, the infimum exists.
[96] The latter is true because if $r^L(p_{inf,1}) = 0$, then $D(p_{inf,1}) \leqslant -q$.

$r^L(\tilde{p}_K) > 0$, and $\ln\left(\frac{\tilde{p}^s_K}{1-\tilde{p}^s_K}\right) - \ln\left(\frac{p^s_{inf,1}}{1-p^s_{inf,1}}\right) < \varepsilon$. As $\tau(\tilde{p}_K) = +\infty$ and $r^L(\tilde{p}_K) > 0$, we must have $\tau(f(\tilde{p}_K)) = +\infty$, and therefore $p_{inf,2} \leqslant f(\tilde{p}_K)$. Finally, by Lemma 12 we then have $\ln\left(\frac{f^s(\tilde{p}_K)}{1-f^s(\tilde{p}_K)}\right) - \ln\left(\frac{\tilde{p}^s_K}{1-\tilde{p}^s_K}\right) \leqslant \ln\left(1-\frac{3q}{4}\right)$. It is then true that

$$\begin{aligned}\ln\left(\frac{p^s_{inf,1}}{1-p^s_{inf,1}}\right) - \ln\left(\frac{p^s_{inf,2}}{1-p^s_{inf,2}}\right) &> \ln\left(\frac{\tilde{p}^s_K}{1-\tilde{p}^s_K}\right) - \varepsilon - \ln\left(\frac{f^s(\tilde{p}_K)}{1-f^s(\tilde{p}_K)}\right)\\ &\geqslant -\ln\left(1-\frac{3q}{4}\right) - \varepsilon.\end{aligned}$$

The last term is greater than $-\ln\left(1-\frac{q}{2}\right)$ for sufficiently small ε.

Case 3 $D(p_{inf,1}) < 0$ and there exists a sequence $\{\tilde{p}^s_k\}$ such that $\tilde{p}^s_k \uparrow p^s_{inf,1}$ and $k < \tau(\tilde{p}_k) < +\infty$, where $\tilde{p}_k := (\tilde{p}^n, \tilde{p}^s_k)$. As $\tau(p_{inf,1}) = +\infty$ in this sub-case and $\tau(\tilde{p}_k) < +\infty$, for any k there exists $\hat{p}_k = (\tilde{p}^n, \hat{p}^s_k)$ with $\hat{p}^s_k \in [\tilde{p}^s_k, p^s_{inf,1}]$ such that $\tau(\hat{p}_k) > k$ and $D(\hat{p}_k) > -\frac{1}{k}$. Note that $\hat{p}_k \to p_{inf,1}$ as $k \to +\infty$. Now suppose (57) does not hold. Fix some arbitrary $\delta > 0$. For any $\delta > 0$ we have $\tau(f^n(\tilde{p}^n), p^s_{inf,2} - \delta)) < +\infty$, so we can find $k > \frac{4}{q}$ such that $\tau(\hat{p}_k) > \tau(f^n(\tilde{p}^n), p^s_{inf,2} - \delta))$. By Lemma 12 we know that $\ln\left(\frac{f^s(\hat{p}_k)}{1-f^s(\hat{p}_k)}\right) - \ln\left(\frac{\hat{p}^s_k}{1-\hat{p}^s_k}\right) \leqslant \ln\left(1-\frac{3q}{4}\right)$. As $r^L(\hat{p}^s_k) > 0$, we must have $\tau(f(\hat{p}_k)) \geqslant \tau(\hat{p}_k)$, and therefore by the monotonicity of $\tau(f^n(\tilde{p}^n), p^s)$ in its second argument we must have $f^s(\hat{p}_k) > p^s_{inf,2} - \delta$. However,

$$\ln\left(\frac{f^s(\hat{p}_k)}{1-f^s(\hat{p}_k)}\right) - \ln\left(\frac{p^s_{inf,2}}{1-p^s_{inf,2}}\right) < \ln\left(1-\frac{3q}{4}\right) - \ln\left(1-\frac{q}{2}\right) < 0,$$

which implies that $f^s(\hat{p}_k) < p^s_{inf,2}$, and by taking sufficiently small δ we achieve a contradiction. □

Having shown (57), consider the sequence $\{p_{inf,k}\}$ where $p_{inf,k} := \left((f^n)^{k-1}(\tilde{p}^n), p^s_{inf,k}\right)$ and $p^s_{inf,k} = \inf\{p^s \mid \tau\left((f^n)^{k-1}(\tilde{p}^n), p^s\right) = +\infty\}$. Equation (57) then implies that $p^s_{inf,k} < \bar{p}$ for all $k > M := \left\lceil \frac{\ln\left(\frac{p^s_{inf,1}}{1-p^s_{inf,1}}\right)}{\ln(1-\frac{q}{2})} \right\rceil$, i.e., we have $p_{inf,k} \in \mathcal{B}^{\downarrow}_0$, and there exists $\varepsilon_k > 0$ such that $p \in \mathcal{B}^{\downarrow}_0$ for all $p = (p^n_{inf,k}, p^s)$ with $p^s \in [p^s_{inf,k}, p^s_{inf,k} + \varepsilon_k)$. However, by definition we have $\tau(p) = 0$ for all $p \in \mathcal{B}^{\downarrow}_0$, which brings us to a contradiction with the definition of $p_{inf,M}$.

Proof of Lemma 2 for $\mathcal{B}^{\uparrow}_0$. Proofs for this and other regions proceed by contradiction: we show that the low-type seller can neither have strict preference towards revealing a review ($r^L(p) = 1$), nor towards deleting a review ($r^L(p) = 0$).

Suppose that at some $p \in \mathcal{B}_0^{\uparrow} \cap \mathcal{R}$ a low-type seller strictly prefers to reveal a bad review, i.e., $r^L(p) = 1$. Then $D(p) \geqslant 0$ and $\tau(p) = +\infty$, which contradicts Lemma 1 for $\mathcal{B}_0^{\uparrow}$. If $r^L(p) = 0$ and $r^H(p) > 0$ instead, then revealing a bad review brings the maximal continuation profit to a low-type seller, while deleting it yields strictly less if no new bad review arrives in time $\tau(p)$, which is finite by Lemma 1 for all $p \in \mathcal{B}_0^{\uparrow}$, so the probability of this happening is strictly positive. That contradicts $r^L(p) = 0$. As $r^L(p) < 1$ for all $p \in \mathcal{B}_0^{\uparrow}$, we have that a low-type seller weakly prefers to conceal a bad review at every state in $\mathcal{B}_0^{\uparrow}$. Therefore, the value of a low-type seller is equal to the value he gets by deleting all further bad reviews: $V^L(p) = \int_0^{\tau(p)} e^{-rt} \cdot \mu dt$. As $V^L(p) = V^L(f(p))$, we must then have $\tau(p) = \tau(f(p))$. □

For further proofs we introduce a new object: the *average drift* at state $p = (p^n, p^s)$ is defined as

$$\bar{D}(p) := \frac{1}{\lambda \pi(p) \tau(p)} \left(\ln\left(\frac{p^s}{1-p^s}\right) - \ln\left(\frac{\bar{p}}{1-\bar{p}}\right) \right).$$

By Lemma 11 $\tau(p)$ is differentiable in p^s, and by Lemma 1 for $\mathcal{B}_0^{\uparrow}$, in any equilibrium $\tau(p) < +\infty$ for all $p \in \mathcal{B}_0^{\uparrow}$. Therefore, in any equilibrium $\bar{D}(p)$ is well defined in $\mathcal{B}_0^{\uparrow}$ and is differentiable with respect to p^s for any $p^s < 1$. Lemma 2 for $\mathcal{B}_0^{\uparrow}$ also states that $\tau(p) = \tau(f(p))$, and therefore function $J(p)$ is well-defined for all $p \in \mathcal{B}_0^{\uparrow}$ by Lemma 11.

Lemma 13. *1. Suppose there exists a state $\tilde{p} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}_0^{\uparrow}$ such that $\bar{D}(\tilde{p}) \leqslant -q - \varepsilon$ for some $\varepsilon \in (0, 1-q]$. Then there exists $\hat{p}^s \in [\bar{p}, \tilde{p}^s]$ such that $\bar{D}(\tilde{p}^n, \hat{p}^s) \leqslant -q - \varepsilon$ and $J(\tilde{p}^n, \hat{p}^s) \geqslant \ln(1+\varepsilon)$.*

2. Suppose there exists a state $\tilde{p} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}_0^{\uparrow}$ such that $\bar{D}(\tilde{p}) \geqslant -q + \varepsilon$ for some $\varepsilon \in (0, q)$. Then there exists $\hat{p}^s \in [\bar{p}, \tilde{p}^s]$ such that $\bar{D}(\tilde{p}^n, \hat{p}^s) \geqslant -q + \varepsilon$ and $J(\tilde{p}^n, \hat{p}^s) \leqslant \ln(1-\varepsilon)$.

Proof. We only show the first statement; the second is proved analogously. Consider a set $S := \{p^s \in [\bar{p}, \tilde{p}^s] \mid J(\tilde{p}^n, p^s) \geqslant \ln(1+\varepsilon)\}$. First, it is nonempty, as otherwise by Lemma 12 we have $D(p) > -q - \varepsilon$ for all p with $p^s \in [\bar{p}, \tilde{p}^s]$, which violates $\bar{D}(\tilde{p}) \leqslant -q - \varepsilon$.[97] Second, S is closed (as $J(p)$ is continuous in p^s) so its upper contour sets are closed in p^s. Finally, S is trivially bounded from above by $\tilde{p}^s$. Therefore, there exists $\hat{p}^s := \sup S \in S$. Moreover, for all $p^s > \hat{p}^s$ we have $J(\tilde{p}^n, p^s) < \ln(1+\varepsilon)$ and, therefore, $D(\tilde{p}^n, p^s) > -q - \varepsilon$, which implies $\bar{D}(\tilde{p}^n, \hat{p}^s) \leqslant -q - \varepsilon$. The second property of $\hat{p}^s$ follows directly from the definition of S. □

[97] If $D(p) \neq -q$, then $p \in \mathcal{R}$ and (12) imply $J(p) = \ln\left(\frac{f^s(p)}{1-f^s(p)}\right) - \ln\left(\frac{p^s}{1-p^s}\right)$.

Proof of Proposition 10. First note that any strategy profile that generates $f^s(p) = p^s$ for all $p \in \mathcal{B}_0^{\uparrow} \cap \mathcal{R}$ constitutes an equilibrium. Indeed, by Lemma 10 $f^s(p) = p^s$ implies $D(p) = -q$ for all p, and therefore $\tau(p) = \tau(f(p))$ for all $p \in \mathcal{B}_0^{\uparrow}$, making both types of sellers indifferent between disclosing and concealing a bad review.

Proof of the converse is split into two steps. In step 1 below we show that if there exists $p \in \mathcal{B}_0^{\uparrow} \cap \mathcal{R}$ such that $J(p) \neq 0$, then there exists a state $\tilde{p}$ such that either $\bar{D}(\tilde{p}) \leqslant -q - \varepsilon$ and $J(\tilde{p}) \geqslant \ln(1+\varepsilon)$, or $\bar{D}(\tilde{p}) \geqslant -q+\varepsilon$ and $J(\tilde{p}) \leqslant \ln(1-\varepsilon)$.[98] Then in step 2 we achieve a contradiction in both of these cases.

Step 1 Suppose there exists $p \in \mathcal{B}_0^{\uparrow}$ such that $J(p) \neq 0$. If $\bar{D}(p) \neq -q$, then the claim is valid by Lemma 13. Now suppose that $\bar{D}(p) = -q$. Then as $J(p) \neq 0$, it must be that $\bar{D}(f(p)) \neq -q$ and we can apply Lemma 13 to $f(p)$.

Step 2 Suppose there exists p_1 such that $\bar{D}(p_1) \leqslant -q - \varepsilon$ and $J(p_1) \geqslant \ln(1 + \varepsilon)$. Denote $K := 1 + \ln(1 + \varepsilon) \cdot \tau(p_1)^{-1}$. As $\tau(f(p_1)) = \tau(p_1)$ and $D(p) \geqslant -1$, it must be that $\bar{D}(f(p_1)) \leqslant K \cdot (-q - \varepsilon)$. Then by Lemma 13 there exists $p_2 = (p_2^s, p_2^n)$ with $p_2^s \in [\bar{p}, f^s(p_1)]$ and $p_2^n := f^n(p_1^n)$ such that $\bar{D}(p_2) \leqslant K(-q-\varepsilon)$ and $J(p_2) \geqslant \ln(1+K(q+\varepsilon)-q) > \ln(1+\varepsilon)$. Iterating this procedure $M := [-\log_K(q + \varepsilon)] + 1$ times we arrive at a state p_M such that $\bar{D}(p_M) \leqslant K^M(-q - \varepsilon) < -1$, which is impossible.

Alternatively, suppose there exists p_1 such that $\bar{D}(p_1) \geqslant -q + \varepsilon$ and $J(p_1) \leqslant \ln(1 - \varepsilon)$. Then as $\tau(f(p_1)) = \tau(p_1)$ and $J(p_1) < 0$, it must be that $\bar{D}(f(p_1)) > -q + \varepsilon$. Then by Lemma 13 there exists $p_2 = (p_2^s, p_2^n)$ with $p_2^s \in [\bar{p}, f^s(p_1)]$ and $p_2^n := f^n(p_1^n)$ such that $\bar{D}(p_2) \geqslant -q + \varepsilon$ and $J(p_2) \leqslant \ln(1 - \varepsilon)$. At the same time, $\ln\left(\frac{p_2^s}{1-p_2^s}\right) - \ln\left(\frac{p_1^s}{1-p_1^s}\right) < \ln(1 - \varepsilon)$. Iterating this procedure $M := \left[\left(\ln\left(\frac{\bar{p}}{1-\bar{p}}\right) - \ln\left(\frac{p_1^s}{1-p_1^s}\right)\right) \cdot \frac{1}{\ln(1-\varepsilon)}\right] + 1$ times we achieve a state $p_M = (p_M^s, p_M^n)$ such that $p_M^s < \bar{p}$ and $\tau(p_M) = \tau(p_1)$, – a contradiction. □

Proof of Corollary 2. Proposition 10 and Lemma 10 imply that $D(p) = -q$ for all $p \in \mathcal{B}_0$ in any equilibrium. Therefore, (77) states that $\tau(p)$ for any given p must be the same in any equilibrium. Representation (15) then implies that the same is true for $V^L(p)$. The high type's value $V^H(p)$ is

[98]Note that $J(p) = 0$ implies $f^s(p) = p^s$.

also the same in any equilibrium, since it can be written for $p \in \mathcal{B}_0$ as

$$V^H(p) = \int_0^{\tau(p)} e^{-rt}\left(\mu + (1-\mu)\cdot(1-e^{-\lambda q\mu t})\right)dt + \int_{\tau(p)}^{+\infty} e^{-rt}\left(1-e^{-\lambda q\mu\tau(p)}\right)dt$$
$$= \frac{\mu\,(r+\lambda q)}{r(r+\lambda q\mu)}\cdot\left(1-e^{-(r+\lambda q\mu)\tau(p)}\right).$$

Finally, consumers' behavior and, hence, payoffs are always the same at a given p in any equilibrium. Therefore, for a given $p \in \mathcal{B}_0$ all players' payoffs are the same in any continuation equilibrium. □

Lemma 14. *If $\mu < 1/2$, then $\mathcal{B}_1 \cap \mathcal{R} = \emptyset$.*

Proof. If $p \in \mathcal{B}_1$ then the low-type seller can guarantee himself

$$V^L(p) \geqslant \frac{1-\mu}{r} + (1-e^{-r\tau(p)})\cdot\frac{\mu}{r} > \frac{1-\mu}{r}$$

by deleting all future reviews and retaining naive consumers forever and sophisticated consumers for time $\tau(p)$. Disclosing any bad review makes naive consumers quit the market until a good review arrives (which is never for a low-type seller), so

$$V^L(f(p)) = (1-e^{-r\tau(f(p))})\cdot\frac{\mu}{r} \leqslant \frac{\mu}{r}.$$

As one can see, if $\mu < 1/2$, then $V^L(f(p)) < V^L(p)$, hence the low-type seller is never willing to disclose a bad review. □

Proof of Lemma 2 for $\mathcal{B}_1$. Whenever $\mu < 1/2$, by Lemma 14 we have $\mathcal{B}_1 \cap \mathcal{R} = \emptyset$ so the statement is trivially true. Thus from now on assume $\mu \geqslant 1/2$. We divide the proof into two parts corresponding to two subregions of $\mathcal{B}_1$.

Case 1: $p \in \mathcal{B}_1^{\downarrow}$ There it must be that $(1-q)\cdot r^H(p) > r^L(p)$, as by sacrificing the pool of naive consumers any seller must gain the pool of sophisticated consumers for at least some period of time, so $f^s(p) > \bar{p} > p^s$. In particular, this implies that $r^L(p) = 1$ is not possible in any equilibrium.

As for the second case, suppose instead that $r^L(p) = 0$ and $r^H(p) > 0$. Then any single bad review reveals a high-type seller and trades off the pool of naive consumers for the whole pool of sophisticated consumers forever. Either group under the respective scenario stays on the marker forever, and the other group joins after a good review. Thus $r^L(p) = 0$ is optimal for the low-type seller only if $\mu = 1/2$. In that case the low-type seller is indifferent between disclosing a bad review and concealing it. If, however, $\mu > 1/2$, then the combination of $r^L(p) = 0$ and $r^H(p) > 0$ is impossible, and thus $r^L(p) > 0$.

Case 2: $p \in \mathcal{B}_1^{\uparrow}$ If $\mu = 1/2$, then a strategy profile constitutes an equilibrium in $\mathcal{B}_1^{\uparrow}$ if and only if $r^H(p) = r^L(p) = 0$ for states with $p^s > \bar{p}$, and $r^L(p^n, \bar{p}) = 0$. At $\bar{p}$ the low-type seller can then retain one and only one of two types of consumers on the market while another is driven out forever, and he is therefore indifferent between revealing and deleting (but in equilibrium deletes all bad reviews).

Thus for the remainder of the proof we assume that $\mu > 1/2$ and consider $p \in \mathcal{B}_1^{\uparrow} \cap \mathcal{R}$. If $r^L(p) = 1$, then $D(p) \geqslant 0$, so staying silent at p gives the maximum possible continuation payoff to any seller. On the other hand, by disclosing at p any seller loses naive consumers for at least some time and therefore gets strictly less, – a contradiction with the optimality of $r^L(p) = 1$.

We are left to show that the low-type seller cannot strictly prefer to delete a bad review. Suppose by way of contradiction that there exists $\tilde{p} \in \mathcal{B}_1^{\uparrow} \cap \mathcal{R}$ such that $r^L(\tilde{p}) = 0$ and $r^H(\tilde{p}) > 0$. Then $f^s(\tilde{p}) = 1$ and $\tau(f(\tilde{p})) = +\infty$. Moreover, non-disclosure is on path for the low type in this and all future states, thus deleting this and all future bad reviews must be weakly better for the low-type seller than disclosing a bad review at $\tilde{p}$ and all further bad reviews afterwards:

$$\int_0^{+\infty} e^{-rt}(1-\mu)dt + \int_0^{\tau(\tilde{p})} e^{-rt}\mu dt \geqslant \int_0^{\tau(f(\tilde{p}))} e^{-rt}\mu dt$$
$$\Leftrightarrow \frac{1-\mu}{\mu} + e^{-r\tau(f(\tilde{p}))} \geqslant e^{-r\tau(\tilde{p})}. \tag{58}$$

On the other hand, the high-type seller's value from disclosing a bad review at $\tilde{p}$ is

$$V^H(f(\tilde{p})) = \int_0^{\tau(f(\tilde{p}))} e^{-rt}\left(\mu + (1-\mu)\cdot(1-e^{-\lambda q\mu t})\right)dt + \tag{59}$$
$$+ \int_{\tau(f(\tilde{p}))}^{+\infty} e^{-rt}\left(1 - e^{-\lambda q\mu\tau(f(\tilde{p}))}\right)dt$$
$$= \frac{\mu(r+\lambda q)}{r(r+\lambda q\mu)}\cdot\left(1 - e^{-(r+\lambda q\mu)\tau(f(\tilde{p}))}\right). \tag{60}$$

The value that the high-type seller gets from deleting a bad review at $\tilde{p}$ is at least as large as the value from deleting all bad reviews from $\tilde{p}$ onwards:

$$\begin{aligned} V^H(\tilde{p}) &\geqslant \int_0^{\tau(\tilde{p})} e^{-rt} dt + \int_{\tau(\tilde{p})}^{+\infty} e^{-rt} \left(1 - \mu e^{-\lambda q(\mu\tau(\tilde{p}) + (1-\mu)t)}\right) dt \\ &= \frac{1}{r} \cdot \left(1 - \frac{r\mu}{r + \lambda q (1-\mu)} e^{-(r+\lambda q)\tau(\tilde{p})}\right) \\ &\geqslant \frac{1}{r} - \frac{\mu}{r + \lambda q (1-\mu)} \cdot \left(\frac{1-\mu}{\mu}\right)^{1+\frac{\lambda q}{r}}, \end{aligned}$$

where the last inequality follows from (58) after recalling that $\tau(f(\tilde{p})) = +\infty$. As $r^H(\tilde{p}) > 0$, it must be that $V^H(f(\tilde{p})) \geqslant V^H(\tilde{p})$, which implies:

$$\begin{aligned} \frac{\mu(r+\lambda q)}{r(r+\lambda q\mu)} - \frac{1}{r} + \frac{\mu}{r + \lambda q(1-\mu)} \cdot \left(\frac{1-\mu}{\mu}\right)^{1+\frac{\lambda q}{r}} &\geqslant 0 \\ \Leftrightarrow \frac{1 + \frac{\lambda q}{r}\mu}{1 + \frac{\lambda q}{r}(1-\mu)} \cdot \left(\frac{1-\mu}{\mu}\right)^{\frac{\lambda q}{r}} &\geqslant 1 \end{aligned} \tag{61}$$

Note that (61) holds with equality for $\mu = 1/2$ and that $\left(1 + \frac{\lambda q}{r} x\right)(1-x)^{\frac{\lambda q}{r}}$ is a decreasing function of x for all $x \in (0,1)$. This means that (61) is violated whenever $\mu > 1/2$, so there does not exist any $\tilde{p} \in \mathcal{B}_1^{\uparrow} \cap \mathcal{R}$ with $r^L(\tilde{p}) = 0$.

Finally, as $r^L(p) < 1$ for all $p \in \mathcal{B}_1$ and the low-type seller is indifferent between disclosing and concealing a bad review at all $p \in \mathcal{B}_0^{\uparrow} \cup \mathcal{B}_1$, (58) holds with equality for all $p \in \mathcal{B}_1^{\uparrow}$. □

Proof of Lemma 1 for $\mathcal{B}_{1+}^{\uparrow}$. We prove the claim only for $\mathcal{B}_1^{\uparrow}$. Induction to all further bands is straightforward. Assume the contrary. Then there exists $\tilde{p} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}_1^{\uparrow}$ with $\tau(\tilde{p}) = +\infty$. Consider a state $\tilde{p}_{inf} := (\tilde{p}^n, \tilde{p}^s_{inf}) \in \mathcal{B}_1^{\uparrow}$ where $\tilde{p}^s_{inf} = \inf\{p^s \mid \tau(\tilde{p}^n, p^s) = +\infty\}$. According to Lemma 11, there can be three sub-cases. Either $D(\tilde{p}_{inf}) \geqslant 0$, or $D(\tilde{p}_{inf}) < 0$ and there exists a sequence $\tilde{p}^s_k$ converging to $\tilde{p}^s_{inf}$ either from below or from above such that $D(\tilde{p}^n, \tilde{p}^s_k) \to 0$.

If $D(\tilde{p}_{inf}) \geqslant 0$ or $\tilde{p}^s_k$ converges to $\tilde{p}^s_{inf}$ from above, there exists $\hat{p}$ such that $\tau(\hat{p}) = +\infty$ (i.e., no disclosure at $\hat{p}$ grants the maximal continuation payoff) and $D(\hat{p}) > -q$, with the latter implying that $\hat{p} \in \mathcal{R}$. By deleting all bad reviews the seller can retain both groups of consumers in the market forever starting from $\hat{p}$. However, we know that $V^{\theta}(f(\hat{p}))$ is strictly smaller than the maximal possible payoff for seller of type θ, since this is true for any $p \in \mathcal{B}_0$ with $p^s < 1$. Revealing a bad review at $\hat{p}$ is thus strictly suboptimal for either type of the seller, which contradicts $\hat{p} \in \mathcal{R}$.

If $\tilde{p}^s_k$ converges to $\tilde{p}^s_{inf}$ from below, then for any $\varepsilon > 0$ and any $C > 0$ there exists $\hat{p}$ such that

$D(\hat{p}) > -\varepsilon$ and $\tau(\hat{p}) > C$. The latter property is satisfiable, as improper integral in (77) diverges, and therefore for any $C > 0$ there exists some k such that $\tau(\tilde{p}^n, \tilde{p}^s_{inf} - \frac{1}{k}) > C$. As for the former, we know that $\tau(\tilde{p}^n, \tilde{p}^s_{inf}) - \tau(\tilde{p}^n, \tilde{p}^s_{inf} - \frac{1}{k}) = +\infty$, and therefore there exists $\hat{p}^s \in [\tilde{p}^s_{inf} - \frac{1}{k}, \tilde{p}^s_{inf}]$ such that $D(\tilde{p}^n, \hat{p}^s) > -\varepsilon$. As the seller's value $V^\theta(p)$ in any state $p \in \mathcal{B}_0^\uparrow$ with $p^s < 1$ is strictly less than the maximal one and as C can be made arbitrarily large, we can find C large enough that the value of disclosure is strictly less than the value of staying silent. Since $\hat{p} \in \mathcal{R}$ as long as $\varepsilon < q$, we achieve a contradiction. □

Proof of Proposition 11. It is shown in Lemma 14 that if $\mu < 1/2$, then $r^L(p) = r^H(p) = 0$ for all $p \in \mathcal{B}_1 \cap \mathcal{R}$ is the only possible equilibrium strategy profile. To show the second condition, recall from Lemma 2 that a low-type seller must be indifferent between revealing a bad review at $p \in \mathcal{B}_1^\uparrow$ and not, and that his indifference condition can be written as

$$\frac{1-\mu}{\mu} + e^{-r\tau(f(p))} = e^{-r\tau(p)}. \tag{62}$$

As $\tau(f(p)) \leqslant +\infty$, it should be that $\tau(p) \leqslant \frac{1}{r} \ln \frac{\mu}{1-\mu}$. Therefore, as $D(p) \geqslant -1$, we have $\ln \frac{p^*}{1-p^*} - \ln \frac{\bar{p}}{1-\bar{p}} \leqslant \frac{\lambda}{r} \ln \frac{\mu}{1-\mu}$ which gives the result. □

Proof of Proposition 12. The claim was already established for $\mathcal{B}_1^\downarrow$ in the proof of Lemma 2 for $\mathcal{B}_1$. We are left to show it for $\mathcal{B}_1^\uparrow$. As Lemma 11 shows, we can construct a mapping g such that

$$\frac{1-\mu}{\mu} + e^{-r\tau(g(p))} = e^{-r\tau(p)}, \tag{63}$$

so $g(p) = f^s(p)$ for all $p \in \mathcal{R}$. Further, $g(p)$ can be represented as

$$\ln\left(\frac{g(p)}{1-g(p)}\right) = J(p) + \ln\left(\frac{p^s}{1-p^s}\right) \tag{64}$$

for some function $J(p)$ which is differentiable in p^s. Taking the derivative of both sides of (63) with respect to p^s, we get

$$e^{-r\tau(p)} \cdot \frac{d\tau(p)}{dp^s} = e^{-r\tau(g(p))} \cdot \frac{d\tau(g(p))}{dg(p)} \frac{dg(p)}{dp^s}. \tag{65}$$

As is shown in Lemma 11, $\frac{d\tau(p)}{dp^s} = \left(\lambda p^s (1-p^s)\, \pi(p) D(p)\right)^{-1}$. Differentiating (64) we get

$$\frac{dg(p)}{dp^s} = \frac{e^{-J(p)} + p^s(1-p^s)\frac{dJ(p)}{dp^s}e^{-J(p)}}{\left[p^s + (1-p^s)e^{-J(p)}\right]^2}.$$

Plugging the three derivatives, we get that (65) corresponds to

$$e^{-r\tau(p)} \cdot \mu q = e^{-r\tau(g(p))} \cdot (-D(p)) \cdot \left[1 + p^s(1-p^s) \cdot \frac{dJ(p)}{dp^s}\right].$$

Plugging (63) into the expression above we get

$$(-D(p)) \cdot \left[1 + p^s(1-p^s) \cdot \frac{dJ(p)}{dp^s}\right] = q \cdot \left[\mu + (1-\mu) \cdot e^{r\tau(g(p))}\right]. \tag{66}$$

Consider state $(p^n, \bar{p}) \in \mathcal{B}_1$. We know $\tau(p^n, \bar{p}) = 0$, therefore (63) implies $\tau(g(p^n, \bar{p})) > 0$, which in turn means $J(p^n, \bar{p}) > 0$. For any $p \in \mathcal{B}_1^{\uparrow}$ we have $\tau(g(p)) > \tau(g(p^n, \bar{p})) > 0$, therefore there exists $\varepsilon > 0$ such that the RHS of (66) is larger than $q + \varepsilon$. If additionally $\frac{dJ(p)}{dp^s} < 0$, (66) implies $D(p) < -q - \varepsilon$, and consequently $J(p) \geqslant \ln(1+\varepsilon)$ by Lemma 12. It then follows from continuity of $J(p)$ in p^s that $J(p) > 0$ for all $p \in \mathcal{B}_1^{\uparrow}$. For there to exist $p \in \mathcal{B}_1^{\uparrow}$ such that $J(p) \leqslant 0$ there should exist $\tilde{p}$ such that $J(\tilde{p}) \in (0, \ln(1+\varepsilon))$ and $\frac{dJ(\tilde{p})}{dp^s} < 0$, which is ruled out by the argument above. □

Lemma 15. *If $\mu \geqslant 1/2$, then for any set $\tilde{\mathcal{R}} \subseteq \mathcal{B}_1^{\downarrow}$ there exists an equilibrium with $\mathcal{B}_1^{\downarrow} \cap \mathcal{R} = \tilde{\mathcal{R}}$.*

Proof. As Lemma 2 states, for $\mu \geqslant 1/2$ the low-type seller is indifferent between disclosing a bad review and concealing it at all $p \in \mathcal{B}_1^{\downarrow} \cap \mathcal{R}$. This indifference is given by (62), and using $\tau(p) = 0$ for all $p \in \mathcal{B}_1^{\downarrow} \cap \mathcal{R}$ as well as the fact that $\tau(f(p)) = \frac{1}{\lambda q \mu}\left[\ln\left(\frac{f^s(p)}{1-f^s(p)}\right) - \ln\left(\frac{\bar{p}}{1-\bar{p}}\right)\right]$ it can be rewritten as[99]

$$\left(\frac{\bar{p}}{1-\bar{p}} \cdot \frac{1-f^s(p)}{f^s(p)}\right)^{\frac{r}{\lambda q \mu}} = 2 - \frac{1}{\mu}$$

$$\Leftrightarrow \frac{f^s(p)}{1-f^s(p)} = \frac{p^s}{1-p^s} \cdot \frac{(1-q)\cdot r^H(p)}{r^L(p)} = \frac{\bar{p}}{1-\bar{p}}\left(2 - \frac{1}{\mu}\right)^{-\frac{\lambda q \mu}{r}}.$$

Next we consider incentives of a high-type seller. Since $r^H(p) > 0$, he should weakly prefer to reveal a bad review. We further show that this is always true whenever $\mu \geqslant 1/2$ (and the preference is strict if $\mu > 1/2$), and therefore $r^H(p) = 1$ constitutes an equilibrium in $\mathcal{B}_1^{\downarrow}$. The value from

[99] As $D(p) = -q$ in $\mathcal{B}_0^{\uparrow}$, the expression for $\tau(f(p))$ follows from (77) and the fact that $\pi(p) = \mu$ for $p \in \mathcal{B}_0^{\uparrow}$.

revealing a bad review can be computed by plugging (62) and $\tau(p) = 0$ into (60) to obtain

$$V^H(f(p)) = \left(\frac{1}{r} - \frac{1-\mu}{r+\lambda q\mu}\right) \cdot \left(1 - \left(2 - \frac{1}{\mu}\right)^{1+\frac{\lambda q\mu}{r}}\right). \tag{67}$$

The value of staying silent at p is no greater than supremum over all T of expected payoffs from staying silent until T and then receiving and disclosing a bad review exactly at T (with $T = +\infty$ allowed as an option to stay silent forever). The remainder of this proof shows that this amount is smaller than $V^H(f(p))$, which finalizes the argument. The supremum is equal to

$$\bar{V} = \sup_T \left\{ \int_0^T e^{-rt}\left[1 - \mu \cdot e^{-\lambda q(1-\mu)t}\right] dt + \right.$$
$$\left. + e^{-rT}\left(e^{-\lambda q(1-\mu)T} \cdot V^H(f(p_T)) + \left(1 - e^{-\lambda q(1-\mu)T}\right) \cdot \frac{1}{r}\right)\right\}.$$

By simplifying the expression above we obtain

$$\bar{V} = \sup_T \left(\frac{1}{r} - \frac{\mu}{r+\lambda q(1-\mu)}\right) \cdot \left(1 - e^{-(r+\lambda q(1-\mu))T}\right) + e^{-(r+\lambda q(1-\mu))T} \cdot V^H(f(p_T)),$$

which is a convex combination of $\left(\frac{1}{r} - \frac{\mu}{r+\lambda q(1-\mu)}\right)$ and $V^H(f(p_T))$. The latter is given by (67) and is thus independent of T. Therefore, to finalize the argument we need to show that

$$V^H(f(p)) \geqslant \left(\frac{1}{r} - \frac{\mu}{r+\lambda q(1-\mu)}\right),$$

which would mean that staying silent at p is weakly worse than revealing a bad review at p, and the equality is attained only when $\mu = 1/2$. Indeed, the condition above is equivalent to

$$\frac{1}{1+\frac{\lambda q(1-\mu)}{r}} \cdot \left(2 - \frac{1}{\mu}\right) \geqslant \left(2 - \frac{1}{\mu}\right)^{1+\frac{\lambda q\mu}{r}}. \tag{68}$$

For $\mu = 1/2$ the inequality is trivially satisfied with equality. And for $\mu \in (1/2, 1)$ we have

$$\left(1 - \frac{1-\mu}{\mu}\right)^{\frac{\lambda q\mu}{r}} < e^{-\frac{\lambda q(1-\mu)}{r}} < \frac{1}{1+\frac{\lambda q(1-\mu)}{r}},$$

which concludes the argument. □

Proof of Lemma 2 for $\mathcal{B}_{2+}$. Suppose not: there exists some $p \in \mathcal{R}$ at which the low-type seller has strict preference. Depending on the direction of this preference, two cases are possible:

Case 1: $r^L(p) = 0$, $r^H(p) > 0$ Then $f^s(p) = 1$ and $f^n(p) \geqslant \bar{p}$, so by revealing this bad review and deleting all future ones the seller can guarantee himself the maximal possible continuation payoff. Therefore, deleting bad review at p cannot be strictly better than leaving it – a contradiction.

Case 2: $r^L(p) = 1$, $r^H(p) \leqslant 1$ It implies $D(p) \geqslant 0$. If $p^s \geqslant \bar{p}$, then this contradicts Lemma 1 for $p \in \mathcal{B}^{\uparrow}_{1+}$. If, however, $p^s < \bar{p}$, then by Lemma 10 $D(p) \geqslant 0$ implies that $f^s(p) < p^s$ for bad reviews revealed at p, and therefore $f^s(p) < \bar{p}$. The low-type seller's value from revealing a bad review in $\mathcal{B}^{\downarrow}_1$ is equal to the value of deleting all future bad reviews starting from $f(p)$. Deleting a bad review in $\mathcal{B}^{\downarrow}_2$ can guarantee at least the same value by case of deleting all bad reviews. This means that despite we've assumed $r^L(p) = 1$, the low-type seller is indeed indifferent between disclosure and concealment at p.

We have shown that the low-type seller's value at any $p \in \mathcal{B}_{2+}$ is equal to that from deleting all bad reviews starting from p, and the value of disclosure at p is equal to the value he gets deleting all bad reviews starting from $f^s(p)$. Thus the indifference condition of the low-type seller results in

$$\int_0^{\tau(p)} e^{-rt}dt + (1-\mu)e^{-r\tau(p)}\int_0^{+\infty} e^{-rt}dt = \int_0^{\tau(f(p))} e^{-rt}dt + (1-\mu)e^{-r\tau(f(p))}\int_0^{+\infty} e^{-rt}dt,$$

which can be further reduced to

$$\tau(f(p)) = \tau(p). \tag{69}$$

This concludes the proof. □

Proof of Proposition 13. We show the claim for $\mathcal{B}^{\uparrow}_2$. Induction to $\mathcal{B}^{\uparrow}_k$ with $k > 2$ is straightforward.

As Lemma 11 shows, we can construct mapping g such that $\tau(g(p)) = \tau(p)$, and for some function $J(p)$ which is continuous in p^s we have:

$$\ln\left(\frac{g(p)}{1-g(p)}\right) = J(p) + \ln\left(\frac{p^s}{1-p^s}\right).$$

Now suppose that there exists $\tilde{p} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}^{\uparrow}_2$ such that $f^s(\tilde{p}) < \tilde{p}^s$. Then $g(\tilde{p}) = f^s(\tilde{p})$, and therefore $J(\tilde{p}) < 0$. As $J(p)$ is a continuous function of p^s and $J(\tilde{p}^n, \bar{p}) = 0$, there exists $\hat{p}^s < \tilde{p}^s$ such that $J(\tilde{p}^n, \hat{p}^s) = 0$ and $J(\tilde{p}^n, p^s) < 0$ for all $p^s \in (\hat{p}^s, \tilde{p}^s]$. Thus $g(\tilde{p}^n, p^s) \leqslant p^s$ for all $p^s \in [\hat{p}^s, \tilde{p}^s]$.

Therefore, by Lemmas 10 and 11 we must have $D(\tilde{p}^n, p^s) \geqslant -q$ for all $p^s \in [\hat{p}^s, \tilde{p}^s]$. However, $D(p) \leqslant -q$ for all $p \in \mathcal{B}_1^{\uparrow}$ which violates $\tau(g(\tilde{p})) - \tau(g(\hat{p})) = \tau(\tilde{p}) - \tau(\hat{p})$ given representation (77), where $\hat{p} = (\tilde{p}^n, \hat{p}^s)$. $\square$

Proof of Theorem 2. The statement of the Theorem follows directly from Propositions 10, 12 and 13. $\square$

Proof of Corollary 1. From Theorem 2 and expression (12) we have

$$(1-q) \cdot r^H(p) \geqslant r^L(p).$$

Therefore, if $r^H(p) = 0$, then we must have $r^L(p) = 0$ and $p \notin \mathcal{R}$. If $r^H(p) > 0$, then

$$r^H(p) > (1-q) \cdot r^H(p) \geqslant r^L(p)$$

which proves the claim. $\square$

Proof of Theorem 3. To prove the first part note that first, by Corollary 2 all continuation equilibria are payoff-equivalent in $\mathcal{B}_0^{\uparrow}$. Next, if $\mu < 1/2$, then Lemma 14 implies that $D(p) = -q$ for all $p \in \mathcal{B}_1$, and therefore all continuation equilibria are payoff-equivalent in $\mathcal{B}_1$ as well. As $\mathcal{B}_1^{\downarrow} \cap \mathcal{R} = \emptyset$ by Lemma 14 and p^s can never cross $\bar{p}$ from below, seller's value $V^\theta(p)$ for $p \in \mathcal{B}_{2+}^{\downarrow}$ is equal in any equilibrium to the value of keeping naive consumers in the market forever. Finally, in any equilibrium $D(p) = -q$ for all $p \in \mathcal{B}_{2+}^{\uparrow}$: by Theorem 2 and Lemma 10, $D(p) \leqslant -q$, and if there exists an equilibrium and $p \in \mathcal{B}_{2+}^{\uparrow}$ with $D(p) < -q$, then $J(p) > 0$ by Lemma 10, which violates $\tau(f(p)) = \tau(p)$ as $D(p) = -q$ for all $p \in \mathcal{B}_1$. This implies that $\tau(p)$ is constant across equilibria, which together with the above gives payoff-equivalence in $\mathcal{B}_{2+}^{\uparrow}$.

If $\mu = 1/2$, then $D(p) = -q$ for all $p \in \mathcal{B}_1^{\uparrow} \cap \mathcal{R}$ with $p^s > \bar{p}$. Since on any path of play the game only passes through one state in $\mathcal{B}_1^{\uparrow}$ with $p^s = \bar{p}$ (which is the only state in $\mathcal{B}_1^{\uparrow}$ where $D(p) < -q$ is possible), and drift there is still negative, $\tau(p)$ in any equilibrium must be the same as in case $\mu < 1/2$ (where $\mathcal{B}_1^{\uparrow} \cap \mathcal{R} = \emptyset$) for all $p \in \mathcal{B}_1^{\uparrow}$. The same logic as above can then establish that $D(p) = -q$ for all $p \in \mathcal{B}_{2+}^{\uparrow}$. Finally, in case $\mu = 1/2$ it may be that $\mathcal{B}_1^{\downarrow} \cap \mathcal{R} \neq \emptyset$, but both types of seller are in any such p indifferent between revealing and deleting a bad review, and therefore receive the same payoff as if $\mathcal{B}_1^{\downarrow} \cap \mathcal{R} = \emptyset$.

The remainder of the proof is devoted to constructing an equilibrium that satisfies the requirements of the second part of Theorem 3. We propose a strategy profile and show that it satisfies all equilibrium conditions.

Construct the strategy profile as follows. Let $\mathcal{B}_0^{\uparrow} \cap \mathcal{R} = \emptyset$, and for all $p \in \mathcal{B}_1^{\downarrow}$ build the strategy profile $\left(r^H, r^L\right)$ in such a way that $\mathcal{B}_1^{\downarrow} \cap \mathcal{R} = \mathcal{B}_1^{\downarrow}$, – the latter is possible by Lemma 15.

For $\mu > 1/2$ the inequality in (68) is strict for all $p \in \mathcal{B}_1^{\downarrow}$, so by continuity of preferences of the high-type seller there exists $\varepsilon_1 > 0$ such that he strictly prefers to reveal at all $p \in \{\mathcal{B}_1^{\uparrow} \mid p^s \in [\bar{p}, \bar{p} + \varepsilon_1)\}$, i.e., these states can belong to $\mathcal{R}$ in equilibrium. In all such states it must be that $r^H(p) = 1$, and $r^L(p)$ is then defined implicitly by (62). The latter can be reduced to the following differential equation for $J(p)$:

$$\left(1 - (1-q)e^{-J(p)}\right) \cdot \left[1 + p^s(1-p^s) \cdot \frac{dJ(p)}{dp^s}\right] = q \cdot \left[\mu + (1-\mu) \cdot \left(\frac{1-\bar{p}}{\bar{p}} \cdot \frac{p^s}{1-p^s}\right)^{\frac{r}{\lambda q \mu}} \cdot e^{\frac{r}{\lambda q \mu} J(p)}\right] \tag{70}$$

with an initial condition $J(p^n, \bar{p}) = -\frac{\lambda q \mu}{r} \ln\left(2 - \frac{1}{\mu}\right)$.[100] Then $r^L(p)$ can be obtained from $J(p) = \ln(1-q) - \ln r^L(p)$. By the existence theorem (see Pontryagin [1962], chapter 4, §21) a solution to (70) exists in some neighborhood of $(p^n, \bar{p})$, i.e., there exists $\varepsilon_2 > 0$ such that $J(p)$, and consequently $r^L(p)$, is well-defined for all $p = (p^n, p^s)$ with $p^s \in [\bar{p}, \bar{p}+\varepsilon_2)$. Take $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ and set $r^L(p)$ for all $p \in \{\mathcal{B}_1^{\uparrow} \mid p^s < \bar{p} + \varepsilon\}$ as prescribed by the procedure above. At the remaining states $p \in \{\mathcal{B}_1^{\uparrow} \mid p^s \geqslant \bar{p} + \varepsilon\}$ set $r^H(p) = r^L(p) = 0$.

The strategy profile in $\mathcal{B}_{2+}$ is constructed as follows. For any $p \in \mathcal{B}_{2+}^{\downarrow}$ let $r^H(p) = 1$ and $r^L(p) = (1-q) \cdot \left(\frac{p^s}{1-p^s} \cdot \frac{1-\bar{p}}{\bar{p}}\right)^{\frac{1}{2}}$, which together lead to $J(p) = \frac{1}{2} \cdot \left(\ln\left(\frac{\bar{p}}{1-\bar{p}}\right) - \ln\left(\frac{p^s}{1-p^s}\right)\right) > 0$, meaning $\bar{p} > f^s(p) > p^s$. In $\mathcal{B}_{2+}^{\uparrow}$ let $r^H(p) = r^L(p) = 0$ for $p \in \{\mathcal{B}_{2+}^{\uparrow} \mid p^s = \bar{p}\}$. Let $r^H(p) = 1$ for all $p \in \{\mathcal{B}_{2+}^{\uparrow} \mid p^s > \bar{p}\}$. We compute $r^L(p)$ inductively over bands, where the induction statement is "$r^L(p)$ is constructed for all $p \in \mathcal{B}_k^{\uparrow}$ and it is such that $D(p) \leqslant -q$". This is true by construction for $k = 1$, which starts the induction. Suppose it holds for $k-1$. For $p \in \mathcal{B}_k^{\uparrow}$ we construct $r^L(p)$ so that (69) holds. In particular, consider a change of variable $z = \ln\left(\frac{p^s}{1-p^s}\right)$ and let $J(p^n, z)$ represent, with abuse of notation, the respective transformation of $J(p)$, i.e., $J(p^n, z) = \ln(1-q) - \ln r^L\left(p^n, \frac{e^z}{1+e^z}\right)$. Then taking the derivatives of both sides of (69) with

[100] This initial condition is such that $J(p)$ is continuous at $(p^n, \bar{p})$.

respect to z, we obtain the following differential equation for $J(p^n, z)$:

$$\left(1-(1-q)e^{-J(p^n,z)}\right)\cdot\left[1+\frac{dJ(p^n,z)}{dz}\right]=-D\left(f^n(p),z+J(p^n,z)\right) \tag{71}$$

with the initial condition $J\left(p^n, \ln\left(\frac{\bar{p}}{1-\bar{p}}\right)\right)=0$.[101]

We next show that a solution to (71) exists and is nonnegative for all $z \geqslant \ln\left(\frac{\bar{p}}{1-\bar{p}}\right)$. Suppose that there exists $p=(p^n,p^s)\in\mathcal{B}_k^{\uparrow}$ such that $J\left(p^n, \ln\left(\frac{p^s}{1-p^s}\right)\right)=-\varepsilon<0$. As a solution to an ODE, $J(p^n,z)$ is a continuous function of z. Therefore, there exists $\tilde{p}^s\in(\bar{p},p^s)$ such that $J\left(p^n, \ln\left(\frac{\tilde{p}^s}{1-\tilde{p}^s}\right)\right)=\max\{-\frac{1}{2}\varepsilon, \frac{1}{2}\ln(1-q)\}$. Then

$$\left.\frac{dJ(p^n,z)}{dz}\right|_{z=\ln\left(\frac{\tilde{p}^s}{1-\tilde{p}^s}\right)}=\left.\frac{-D(f^n(p),z+J(p^n,z))}{1-(1-q)e^{-J(p^n,z)}}\right|_{z=\ln\left(\frac{\tilde{p}^s}{1-\tilde{p}^s}\right)}-1>\frac{q}{q}-1=0.$$

Therefore, as we increase z from $\ln\left(\frac{\tilde{p}^s}{1-\tilde{p}^s}\right)$, $J(p^n,z)$ could never fall below $-\frac{\varepsilon}{2}$, while we have assumed $J\left(p^n, \ln\left(\frac{p^s}{1-p^s}\right)\right)=-\varepsilon$ – a contradiction. As $\varepsilon>0$ was taken arbitrarily, it shows that $J(p^n,z)\geqslant 0$ for all $z\geqslant\ln\left(\frac{\bar{p}}{1-\bar{p}}\right)$. We next can take arbitrary solution to (71) in the neighborhood of its initial condition, the existence of which is ensured by the existence theorem (see Pontryagin [1962], chapter 4, §21). It can be extended for all $z\geqslant\ln\left(\frac{\bar{p}}{1-\bar{p}}\right)$ if and only if $J(p^n,z)<+\infty$ for all such z (see Pontryagin [1962], chapter 4, §24) which is true as

$$\left|\frac{dJ(p^n,z)}{dz}\right|<\frac{1}{q}-1=\frac{1-q}{q}<+\infty.$$

Consequently, by Lemma 10 we obtain that $D(p)\leqslant -q$ for all $p\in\mathcal{B}_k^{\uparrow}$, which concludes this part of the proof.

We next show that the constructed strategy profile constitutes an equilibrium. We first show that the low-type seller is indifferent whether to reveal a bad review or to conceal it at all $p\in\mathcal{B}_{2+}\cap\mathcal{R}$. If $p\in\mathcal{B}_{2+}^{\downarrow}$, then by construction $0<r^L(p)<1$. From Lemma 15 we also know that $0<r^L(p)<1$ for $p\in\mathcal{B}_1^{\downarrow}\cap\mathcal{R}$. Then the value of a low-type seller in any $p\in\mathcal{B}_{1+}^{\downarrow}\cap\mathcal{R}$ is equal to the value he receives in case he deletes all future bad reviews: $V^L(p)=\frac{1-\mu}{r}$. Therefore, a low-type seller is indeed indifferent between disclosing a bad review and deleting it for any $p\in\mathcal{B}_{2+}^{\downarrow}\cap\mathcal{R}$. For $p\in\mathcal{B}_{1+}^{\uparrow}\cap\mathcal{R}$ the indifference directly follows from the way $r^L(p)$ is constructed and the fact that $r^L(p)<1$.[102]

[101] The RHS of (71) is not smooth in $J(p^n,z)$, but is piecewise smooth. Therefore, as a solution to (71) we take a composition of two solutions which are pasted together using continuity.

[102] The latter is true as $J(p)<+\infty$ for all $p\in\mathcal{B}_{1+}^{\uparrow}\cap\mathcal{R}$.

By construction, the high-type seller strictly prefers to reveal bad reviews at all $p \in \mathcal{B}_1 \cap \mathcal{R}$. We proceed by showing that the high-type seller weakly prefers to reveal a bad review at all $p \in \mathcal{B}_{2+}^{\downarrow} \cap \mathcal{R}$. Concealing a review at $p \in \mathcal{B}_2^{\downarrow} \cap \mathcal{R}$ cannot yield him a payoff higher than if he could choose time T at which a bad review will arrive and will be revealed:

$$\begin{aligned}
V^H(p) &\leqslant \max_{T>0} \left\{ \int_0^T e^{-(r+\lambda q(1-\mu))t} \cdot (1-\mu) \cdot \left(1 + \frac{\lambda q}{r}\right) dt + e^{-(r+\lambda q(1-\mu))T} \cdot V^H\left(f\left(p_T\right)\right) \right\} \\
&= \max_{T>0} \left\{ \left(1 - e^{-(r+\lambda q(1-\mu))T}\right) \cdot \left(\frac{1}{r} - \frac{\mu}{r + \lambda q\left(1-\mu\right)}\right) + e^{-(r+\lambda q(1-\mu))T} \cdot V^H\left(f\left(p_T\right)\right) \right\} \\
&\leqslant \max_{T>0} V^H\left(f\left(p_T\right)\right) = V^H\left(f\left(p\right)\right)
\end{aligned}$$

where process p_t is given by (14) with initial condition $p_0 = p$. The last inequality holds because

$$V^H(p) \geqslant \int_0^{+\infty} e^{-rt} \left[e^{-\lambda q(1-\mu)t} \cdot (1-\mu) + \left(1 - e^{-\lambda q(1-\mu)t}\right)\right] dt = \left(\frac{1}{r} - \frac{\mu}{r + \lambda q(1-\mu)}\right)$$

for all $p \in \mathcal{B}_1^{\downarrow}$ since the high-type seller can delete all future bad reviews. The last equality holds because $V^H(f(p_T))$ is independent of T. Indeed, distributions of arrival times of the next buying consumer are the same for all $p \in \mathcal{B}_1^{\downarrow}$. Therefore, $V^H(p)$ is the same for all $p \in \mathcal{B}_1^{\downarrow}$. The resulting inequality $V^H(p) \leqslant V^H(f(p))$ implies that the high-type seller weakly prefers to reveal a bad review at all $p \in \mathcal{B}_2^{\downarrow}$. The argument above can be extended by induction to all further bands in order to obtain that $V^H(p) \leqslant V^H(f(p))$ for all $p \in \mathcal{B}_{2+}^{\downarrow}$.

We are left to show that the high type at least weakly prefers to reveal a bad review in $\mathcal{B}_{2+}^{\uparrow}$. We show the argument for $\mathcal{B}_2^{\uparrow}$, and the argument for $\mathcal{B}_k^{\uparrow}$ with higher k then follows by induction. Fix some state $p = (p^n, p^s) \in \mathcal{B}_2^{\uparrow} \cap \mathcal{R}$. The high-type seller's value in case he decides to conceal a bad review at p is bounded from above by his payoff when he can receive and reveal a bad review at any time T of his choice:

$$\begin{aligned}
V^H(p) \leqslant \max \Bigg\{ &\max_{T \leqslant \tau(p)} \int_0^T e^{-rt} \left(1 + \frac{\lambda q}{r}\right) dt + e^{-rT} \cdot V^H(f(p_T)), \\
&\int_0^{\tau(p)} e^{-rt} \left(1 + \frac{\lambda q}{r}\right) dt + e^{-r\tau(p)} \cdot V^H(p^n, \bar{p}) \Bigg\}, \quad (72)
\end{aligned}$$

where we use that $p^s_{\tau(p)} = \bar{p}$. On the one hand, since deleting all bad reviews is always feasible for the high-type seller, we have

$$V^H(f(p)) \geqslant \int\limits_0^{\tau(f(p))} e^{-rt}\left(1+\frac{\lambda q}{r}\right)dt + e^{-r\tau(f(p))}\cdot V^H(f^n(p),\bar{p})$$
$$\geqslant \int\limits_0^{\tau(p)} e^{-rt}\left(1+\frac{\lambda q}{r}\right)dt + e^{-r\tau(p)}\cdot V^H(p^n,\bar{p})$$

where the second inequality follows because by construction $\tau(p) = \tau(f(p))$, and $V^H(f^n(p),\bar{p}) \geqslant V^H(p^n,\bar{p})$ as shown above.[103] On the other hand, for any $T \leqslant \tau(p)$ we can write

$$V^H(f(p)) \geqslant \int\limits_0^{T} e^{-rt}\left(1+\frac{\lambda q}{r}\right)dt + e^{-rT}\cdot V^H(f(p_T))$$

because the high-type seller can reveal no reviews during $[0,T]$, and because if $p_T \in \mathcal{R}$, then the process given by (14) with starting point $f(p)$ reaches value $f(p_T)$ at exactly time T (this is since $\tau(p) = \tau(f(p))$ for all $p \in \mathcal{B}_2^{\uparrow} \cap \mathcal{R}$), while if $p_T \notin \mathcal{R}$, then $V^H(f(p_T)) = 0$. Everything said above implies that $V^H(f(p)) \geqslant V^H(p)$ for all $p \in \mathcal{B}_{2+}^{\uparrow}$, which concludes the proof that strategy profile is an equilibrium.

Finally, to conclude the proof of Theorem 3 we need to show that the equilibrium has the desired properties. We start with the fact that $f^s(p) > p^s$ for all $p \in \mathcal{R}$. By construction the strategy profile already implies $f^s(p) > p^s$ for all $p \in \left(\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_{2+}^{\downarrow}\right) \cap \mathcal{R}$. We next establish the claim for $p \in \mathcal{B}_2^{\uparrow}$. From (71) we know that $J(p) \geqslant 0$ for all $p \in \mathcal{B}_{2+}^{\uparrow}$. Suppose by way of contradiction that there exists some $\tilde{p} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}_2^{\uparrow}$ such that $f^s(\tilde{p}) = \tilde{p}^s$, i.e., $J(\tilde{p}) = 0$. Assume first that $f(\tilde{p}) \in \mathcal{R}$ which by Lemma 10 implies $D(f(\tilde{p})) < -q$. Then there exists $\varepsilon > 0$ such that $D(f^n(\tilde{p}), p^s) \leqslant -q - \varepsilon$ for all $p^s \in [f^s(\tilde{p}) - \frac{\varepsilon}{4}, f^s(\tilde{p})]$ as, by construction, $D(p)$ is continuous in p^s.[104] At the same time, we have that $J(p) < J(\tilde{p}) + \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$ for all $p = (\tilde{p}^n, p^s)$ with $p^s \in [\tilde{p}^s - \frac{\varepsilon}{4}, \tilde{p}^s]$. By converse of Lemma 12 this implies that $D(p) > -q - \frac{\varepsilon}{2}$ for those p. Therefore $\tau(\tilde{p}) - \tau(\tilde{p}^n, \tilde{p}^s - \frac{\varepsilon}{4}) > \tau(f(\tilde{p})) - \tau(f^n(\tilde{p}), f^s(\tilde{p}) - \frac{\varepsilon}{4})$. Consequently, $J(\tilde{p}^n, \tilde{p}^s - \frac{\varepsilon}{4}) < 0$, – a contradiction. Now assume $f(\tilde{p}) \notin \mathcal{R}$, that is $D(f(\tilde{p})) = -q$. Then (71) can be solved explicitly. Its general solution satisfies

$$(1-q)(z + J(p^n,z)) + q\ln\left(1 - e^{J(p^n,z)}\right) = C \tag{73}$$

[103] Values at the cutoff are equal to respective values under the cutoff since the latter are constant, and total payoff is insensitive to alterations of flow payoff in a single state (i.e., the fact that sophisticated consumers are buying in $p^s = \bar{p}$ does not affect payoffs).

[104] Otherwise there exists a sequence $\{p_k^s\}$ such that $p_k^s \to f^s(\tilde{p})$ and $D(f^n(\tilde{p}), p_k^s) \to -q$ as $k \to +\infty$ which contradicts the continuity of $D(p)$ in $\mathcal{B}_1^{\uparrow}$.

where C is a constant pinned down by the boundary condition for z_0 where $z_0 = \inf\{z \mid f(p^n, z) \notin \mathcal{R}\}$ and $J(p^n, z_0) > 0$ is given as a solution to (71) for $z \in \left[\ln\left(\frac{\bar{p}}{1-\bar{p}}\right), z_0\right]$ with initial condition $J\left(\ln\left(\frac{\bar{p}}{1-\bar{p}}\right)\right) = 0$. Therefore, C is well-defined and finite. As we have assumed $J(\tilde{p}) = 0$ for some $\tilde{p}$, substituting it into (73) we achieve $C = -\infty$, – a contradiction.

All said above shows that $J(p) > 0$ for all $p \in \mathcal{B}_2^{\uparrow}$. By Lemma 10 it implies $D(p) < -q$ for all $p \in \mathcal{B}_2^{\uparrow}$, and the argument then extends to further bands straightforwardly.

To see that this equilibrium is not payoff-equivalent to an equilibrium with $\mathcal{R} = \emptyset$, note that, for instance, the equilibrium constructed above has $D(p) < -q$ for $p \in \{\mathcal{B}_1^{\uparrow} \mid p^s \in [\bar{p}, \bar{p} + \varepsilon)\}$, as opposed to $D(p) = -q$ in the fully censored equilibrium, meaning that $\tau(p)$ is smaller in the former for all $p \in \mathcal{B}_1^{\uparrow}$. Noticing that $\tau(p)$ directly enters the low-type seller's value in $\mathcal{B}_1^{\uparrow}$ concludes the argument. $\square$

Proof of Proposition 14. By Lemma 2 the low-type seller is indifferent between revealing a bad review and deleting it at all $p \in \mathcal{B}_{1+} \cap \mathcal{R}$. Therefore, $V^L(p) = \frac{1-\mu}{r}$ for all $p \in \mathcal{B}_{1+}^{\downarrow}$ irrespective of equilibrium. For $p \in \mathcal{B}_{1+}^{\uparrow}$ we have

$$V^L(p) = \frac{1-\mu}{r} + (1 - e^{-r\tau(p)}) \cdot \frac{\mu}{r} = \frac{1 - \mu e^{-r\tau(p)}}{r}.$$

Therefore, to show the claim we need to establish that larger $\mathcal{R}$ implies pointwise weakly smaller $\tau(p)$. The claim holds for $\mathcal{B}_0^{\uparrow}$ (larger $\mathcal{B}_0^{\uparrow} \cap \mathcal{R}$ has no effect on $\tau(p)$ for $p \in \mathcal{B}_0^{\uparrow}$). Proceed by induction and show that if the claim holds for $\mathcal{B}_{k-1}^{\uparrow}$, then it also holds for $\mathcal{B}_k^{\uparrow}$. For any $p \in \mathcal{B}_k^{\uparrow}$ we show that if $\tau'(p) = \tau''(p)$, then $\frac{d\tau'(p)}{dp} \leqslant \frac{d\tau''(p)}{dp}$ where objects indexed by single and double primes denote respective objects in the two equilibria under consideration with $\mathcal{R}'$ and $\mathcal{R}'' \subset \mathcal{R}'$ respectively. Three cases are possible for every p with $\tau'(p) = \tau''(p)$:

1. If $p \notin \mathcal{R}'$, then $D'(p) = D''(p) = -q$.

2. If $p \in \mathcal{R}' \backslash \mathcal{R}''$, then $D'(p) \leqslant -q = D''(p)$, where the first inequality follows from Theorem 2 and Lemma 10.

3. If $p \in \mathcal{R}''$, then $\tau'(f(p)) \leqslant \tau''(f(p))$ implies that $J'(p) \geqslant J''(p)$, which in turn means that $D'(p) \leqslant D''(p)$ because both equilibria are semi-separating.

Therefore, $D'(p) \leqslant D''(p)$ for all $p \in \mathcal{B}_k^{\uparrow}$. Since $\tau(p^n, \bar{p}) = 0$ for all p^n, (77) implies that $\tau'(\tilde{p}) \leqslant \tau''(\tilde{p})$ for all $p \in \mathcal{B}_k^{\uparrow}$. $\square$

Proof of Proposition 15. As the seller of a high quality product never receives any bad review, after any bad review beliefs jump to $f^s(p) = f^n(p) = 0$ and no future consumers ever buy the product again. Revealing a bad review thus grants the worst continuation payoff, and is therefore strictly dominated by deleting it for any seller who can guarantee non-zero continuation payoff which is true if either $p^n \geqslant \bar{p}$ or $p^s > \bar{p}$. $\square$

Proof of Proposition 16. First let us introduce some extra notation for the general setting. Let $\mathcal{B}_{-1} = \{(p^n, p^s) \in \mathcal{B}_0 \mid (f^n_-)^{-1}(p^n) \geqslant \bar{p}\}$ and $\mathcal{B}_{-k} = \{(p^n, p^s) \mid \big((f^n_-)^{-1}(p), p^s\big) \in \mathcal{B}_{-k+1}\}$ for $k > 1$.[105] By analogy with $\mathcal{B}_k$ for $k > 0$, $\mathcal{B}_{-k}$ measure distance between p^n and $\bar{p}$: if $p \in \mathcal{B}_{-k}$ for $k > 0$, then k less bad reviews would be required to bring naive consumers back to the market.

Let us also refresh the expressions for belief updating for the general case. Rational consumers' beliefs are updated in the general setting as:

$$\frac{f^s_+(p)}{1 - f^s_+(p)} = \frac{p^s}{1 - p^s} \cdot \frac{q^H_+ r^H_+(p)}{q^L_+ r^L_+(p)}; \qquad \frac{f^s_-(p)}{1 - f^s_-(p)} = \frac{p^s}{1 - p^s} \cdot \frac{q^H_- r^H_-(p)}{q^L_- r^L_-(p)} \tag{74}$$

after good and bad reviews respectively, and as

$$\dot{p}^s \;=\; \lambda p^s \left(1 - p^s\right) \cdot \left[q^H_+ \left(1 - r^H_+(p)\right) + q^H_- \left(1 - r^H_-(p)\right) - q^L_+ \left(1 - r^L_+(p)\right) - q^L_- \left(1 - r^L_-(p)\right)\right] \tag{75}$$

in the absence of reviews. Naive consumers' reaction to good and bad reviews respectively is given by:

$$\frac{f^s_+(p)}{1 - f^s_+(p)} = \frac{p^s}{1 - p^s} \cdot \frac{q^H_+}{q^L_+}; \qquad \frac{f^s_-(p)}{1 - f^s_-(p)} = \frac{p^s}{1 - p^s} \cdot \frac{q^H_-}{q^L_-}.$$

We construct the equilibrium as follows. For good reviews let $\mathcal{R}_+ = \mathcal{B}^{\uparrow}_{-1}$ and $r^\theta_+(p) = 1$ for either θ and all $p \in \mathcal{R}_+$. For bad reviews let $\mathcal{R}_- = \cup_{k \geqslant 1} \mathcal{B}^{\downarrow}_k$ and $r^H_-(p) = 1$ for all $p \in \mathcal{R}_-$. Let $r^L_-(p)$ for $p \in \mathcal{B}^{\downarrow}_{2+}$ be constructed as in Theorem 3. Finally, $r^L_-(p)$ for $p \in \mathcal{B}^{\downarrow}_1$ is constructed below.

We construct $r^L_-(p)$ for $p \in \mathcal{B}^{\downarrow}_1$ in such a way as to make the low-type seller indifferent between revealing a bad review and not. In such construction, $V^L(p) = \frac{1-\mu}{r}$ for any $p \in \mathcal{B}^{\downarrow}_1$ (and actually all $p \in \mathcal{B}^{\downarrow}_{1+}$ given the remainder of the construction), so deleting all future bad reviews is optimal.

[105] Here function f^n is meant in the sense of $[0, 1] \to [0, 1]$ (i.e., $f^n(p^n)$) since, as we remember, $f^n(p)$ does not depend on p^s.

On the other hand, for any $p \in \mathcal{B}^{\uparrow}_{-1}$ we have

$$V^L(p) = \int\limits_0^{\tau(p)} e^{-rt}\left[e^{-\lambda\mu q^L_+ t}\cdot\mu + \left(1 - e^{-\lambda\mu q^L_+ t}\right)\cdot 1\right]dt + e^{-r\tau(p)}\left(1 - e^{-\lambda\mu q^L_+\tau(p)}\right)\cdot\frac{1}{r}$$
$$= \left(1 - e^{-(r+\lambda\mu q^L_+)\tau(p)}\right)\cdot\left(\frac{1}{r} - \frac{1-\mu}{r+\lambda\mu q^L_+}\right).$$

To clarify, this expression describes payoff from selling to sophisticated consumers until $\tau(p)$ and to all consumers after a good review arrives if this happens before $\tau(p)$. The latter is valid because condition $q^H_+\cdot q^H_- \geqslant q^L_+\cdot q^L_-$ ensures that revealing one additional good review in any $p \in \mathcal{B}^{\uparrow}_{-1}$ brings naive consumers back to the market.

Given the strategies defined above, $D(p) = -(q^H_+ - q^L_+) < 0$ for all $p \in \mathcal{B}^{\uparrow}_{-1}$, hence

$$\tau(p) = \frac{1}{\lambda\mu(q^H_+ - q^L_+)}\left(\ln\left(\frac{p^s}{1-p^s}\right) - \ln\left(\frac{\bar{p}}{1-\bar{p}}\right)\right) < \infty \tag{76}$$

for all $p = (p^n, p^s) \in \mathcal{B}^{\uparrow}_{-1}$. Furthermore, $\tau(p)$ is continuous and strictly increasing in p^s, so $V^L(p)$ is continuous and strictly increasing in p^s as well. Finally, $\tau(p) \to \infty$ as $p^s \to 1$ and $\tau(p) \to 0$ as $p^s \to \bar{p}$, therefore $V^L(p)$ spans the whole interval $\left[0, \frac{1}{r} - \frac{1-\mu}{r+\lambda\mu q^L_+}\right]$ across $p \in \mathcal{B}^{\uparrow}_0$.

Fix some $p \in \mathcal{B}^{\downarrow}_1$. Let $\hat{p} \in \mathcal{B}^{\uparrow}_{-1}$ be such that $V^L(\hat{p}) = \frac{1-\mu}{r}$. It exists for reasons described above: $\frac{1-\mu}{r} < \frac{1}{r} - \frac{1-\mu}{r+\lambda\mu q^L_+}$ whenever $\mu > 1/2$. Finally, let $r^L_-(p)$ for $p \in \mathcal{B}^{\downarrow}_1$ be such that $f_-(p) = (f^n_-(p^n), \hat{p}^s)$ (closed-form expression for $r^L_-(p)$ can be obtained from (74)).

The construction above trivially implies $f^s_-(p) > \bar{p} > p^s$ for all $p \in \mathcal{B}^{\downarrow}_1$. It also generates $f_+(p) > p^s$ for all $p \in \mathcal{R}_+$. Construction in Theorem 3 also implies that $f^s_-(p) > p^s$ for all $p \in \mathcal{B}^{\downarrow}_{2+}$. This verifies the first property in the Proposition. The second property is trivial – $\mathcal{R}_-$ is nonempty for the strategy profile constructed above. Therefore, to conclude the proof we need to verify two things: that the constructed strategy profile constitutes an equilibrium and that this equilibrium is payoff-distinct from fully censored equilibrium in any meaning of the latter.

We start by verifying that the strategy profile above constitutes an equilibrium. First, either type of the seller at least weakly prefers to reveal good reviews at all $p \in \mathcal{R}_+$. This is because $f_+(p) \in \mathcal{B}^{\uparrow}_1$ so $D(f(p)) = 0$ and $\tau(f(p)) = \infty$.[106] Simply speaking, revealing a good review moves seller to an absorbing state in which he can retain both naive and sophisticated consumers in the market forever. This attains the maximal payoff, so is always at least weakly optimal.

[106] In case $q^H_+\cdot q^H_- < q^L_+\cdot q^L_-$ which we do not consider in this proposition, one would need to either ensure that prior p_0 is such that $f^n_+(f_-(p)) \geqslant \bar{p}$ for all $p \in \mathcal{B}^{\downarrow}_1$ on equilibrium path, or to verify that the argument to follow holds even if more than one good review is required to achieve $\mathcal{B}^{\uparrow}_1$ from any $p \in \mathcal{B}^{\uparrow}_{-1}$.

Low-type seller is by construction indifferent between deleting and revealing bad reviews at all $p \in \mathcal{B}_1^{\downarrow}$. This indifference extends to $\mathcal{B}_{2+}^{\downarrow}$. If in any $p \in \mathcal{B}_2^{\downarrow}$ the low-type seller chooses to delete a bad review, he can achieve a payoff of $\frac{1-\mu}{r}$ by deleting all future bad reviews as well. At the same time, revealing a bad review at p (or any future state) grants him $V^L(f(p)) = \frac{1-\mu}{r}$ which is exactly the same payoff. The argument can be iterated further to show that the low type is indifferent at all $\mathcal{B}_{2+}^{\downarrow}$.

The only equilibrium property left to verify is the high type's preference. Suppose that the high-type seller is currently in some state $p \in \mathcal{B}_1^{\downarrow}$. If he deletes all future bad reviews, then his payoff equals $\frac{1-\mu}{r}$. If, however, he has a bad review in hand and reveals it, then he arrives at some $f(p)$ with $f^s(p) = \hat{p}^s$ and receives

$$\begin{aligned}
V^H(f(p)) = & \int_0^{\tau(f(p))} e^{-rt}\left[e^{-\lambda\mu q_+^H t}\cdot\mu + \left(1-e^{-\lambda\mu q_+^H t}\right)\cdot 1\right]dt + e^{-r\tau(f(p))}\left(1-e^{-\lambda\mu q_+^H \tau(f(p))}\right)\frac{1}{r} \\
= & \left(1-e^{-(r+\lambda\mu q_+^H)\tau(f(p))}\right)\cdot\left(\frac{1}{r}-\frac{1-\mu}{r+\lambda\mu q_+^H}\right).
\end{aligned}$$

Given that $q_+^H > q_+^L$ and the low type's indifference requires $\left(1-e^{-(r+\lambda\mu q_+^L)\tau(f(p))}\right)\cdot\left(\frac{1}{r}-\frac{1-\mu}{r+\lambda\mu q_+^L}\right) = \frac{1-\mu}{r}$, trivially $V^H(f(p)) > \frac{1-\mu}{r}$. Doing the usual argument with the high-type seller solving a relaxed problem in which he has a choice of when to reveal the bad review (used in proofs of Lemma 15 and Theorem 3), we can arrive at the conclusion that he strictly prefers to reveal a bad review at p. Using the same argument as in the proof of Theorem 3 we can then show that this strict preference propagates to $\mathcal{B}_{2+}^{\downarrow}$. This concludes the proof that the constructed strategy profile is an equilibrium.

Finally, we want to show that the equilibrium above is payoff-distinct from fully censored equilibrium in either sense of the latter (i.e., where $\mathcal{R}_- = \emptyset$ and $\mathcal{R}_+$ is either same as above, or also empty). In either case it is enough to consider $V^H(p)$ at any $p \in \mathcal{B}_1^{\downarrow}$. In either fully censored equilibrium we have $V^H(p) = \frac{1-\mu}{r}$ because the high-type seller is unable to reveal any reviews. In contrast, in the equilibrium constructed above $V^H(p) > \frac{1-\mu}{r}$ because this inequality is true for all $p \in \mathcal{B}_0^{\uparrow}$ and the high-type seller jumps to $\mathcal{B}_0^{\uparrow}$ from $\mathcal{B}_1^{\downarrow}$ (by receiving and revealing a bad review) with a positive probability in finite time. □

Proof of Proposition 17. We construct the equilibrium in a way analogous to Proposition 16 but accounting for fake reviews. For good reviews let $\mathcal{R}_+ = \mathcal{B}_{-1}^{\uparrow}$ and $r_+^{\theta}(p) = \phi_+^{\theta}(p) = 1$ for either θ and all $p \in \mathcal{R}_+$. For bad reviews let $\mathcal{R}_- = \cup_{k\geqslant 1}\mathcal{B}_k^{\downarrow}$ and $r_-^H(p) = \phi_-^H(p) = 1$ for all $p \in \mathcal{R}_-$. For any $p \in \mathcal{B}_{2+}^{\downarrow}$ let $r_-^L(p)$ and $\phi_-^L(p)$ be an arbitrary solution of the equation $\ln\left(\frac{f_-^s(p)}{1-f_-^s(p)}\right) - \ln\left(\frac{p^s}{1-p^s}\right) = \frac{1}{2}\cdot\left(\ln\left(\frac{\bar{p}}{1-\bar{p}}\right) - \ln\left(\frac{p^s}{1-p^s}\right)\right)$.[107]

Finally, $r_-^L(p)$ and $\phi_-^L(p)$ for $p \in \mathcal{B}_1^{\downarrow}$ are constructed in such a way as to make the low type indifferent between revealing bad reviews and not. Similarly to Proposition 16 in such construction we have $V^L(p) = \frac{1-\mu}{r}$ for any $p \in \mathcal{B}_1^{\uparrow}$, while for any $p \in \mathcal{B}_0^{\uparrow}$: $V^L(p) = \left(1 - e^{-(r+\lambda\mu q_+^L)\tau(p)}\right)\cdot\left(\frac{1}{r} - \frac{1-\mu}{r+\lambda\mu q_+^L}\right)$.

Given the strategies defined above, $D(p) = -(q_+^H - q_+^L) < 0$ for all $p \in \mathcal{B}_{-1}^{\uparrow}$ as in Proposition 16 (since effects of fake positive reviews on $D(p)$ imposed by high and low type cancel each other out). Further, $\frac{f_+^s(p)}{1-f_+^s(p)} = \frac{p^s}{1-p^s}\cdot\frac{\lambda q_+^H+\lambda_\phi}{\lambda q_+^L+\lambda_\phi}$ for all $p \in \mathcal{B}_{-1}^{\uparrow}$, meaning that $f_+^s(p) > p^s$ so $f_+(p) \in \mathcal{B}_1^{\uparrow}$ for all $p \in \mathcal{B}_{-1}^{\uparrow}$.

From here the fact that this strategy profile is an equilibrium and all required equilibrium properties can be verified in exactly the same way as in Proposition 16. $\square$

[107] This is analogous to the construction in Theorem 3. It ensures that $f_-^s(p) > p^s$ and $f_-(p) \in \mathcal{B}^{\downarrow}$.

Appendix for "The Limits of Social Learning"

When talking about current strategy in some public state p_t we always assume that equilibrium is not babbling in this state.

We also introduce the following notation

$$G_p(b) = p \cdot \mu^H(b|p) + (1-p) \cdot \mu^L(b|p).$$

This is a cdf of the distribution of private posterior b as perceived by the consumer with the prior given by public belief p.

We will also use extensively the following representation of messaging strategies.

Definition 9. *A* messaging partition $\Sigma(p_t) = \{I_j\}$ *in public state* p_t *given strategy profile* r *is a (possibly uncountable) collection of disjoint sets* I_j *such that*

1. $\bigcup\limits_j I_j = [0,1]$,
2. *for each* I_j *there exists* $m_j \in \mathcal{M}$ *such that* $r(m_j \mid p_t, b_t) = 1$ *if and only if* $b_t \in I_j$.

Strategy profile r *admits representation with a partition* at state p_t if there exists a respective messaging partition $\Sigma(p_t)$.

Definition 10. *Consider two public beliefs* p'_t *and* $p''_t > p'_t$ *and corresponding messaging partitions* $\Sigma(p'_t) = \{I'_j\}$ *and* $\Sigma(p''_t) = \{I''_j\}$. *Then we call messaging partition* $\Sigma(p''_t)$ *a* parallel shift of $\Sigma(p'_t)$ *if for any* $b'_t \in I'_j$ *we have*

$$b''_t := b'_t + \ln\left(\frac{p''_t}{1-p''_t}\right) - \ln\left(\frac{p'_t}{1-p'_t}\right) \in I''_j.$$

We call $\Sigma(p''_t)$ *a consistent parallel shift of* $\Sigma(p'_t)$ *if*

$$\mathbb{E}[b_t \mid b_t \in I'_j] < \bar{p} \;\Rightarrow\; \mathbb{E}[b_t \mid b_t \in I''_j] < \bar{p}.$$

The first result provides a more convenient representation for value function (21). It shows that any posterior $p \in \mathcal{E}(p_t)$ is fully characterized by two numbers $V^H(p)$ and $V^L(p)$.

Lemma 16. *If* $p \in \mathcal{E}(p_t)$ *then*

$$V(p \mid p_t, b_t) = \theta(b_t) - c + \beta\left(b_t V^H(p) + (1-b_t)\, V^L(p)\right), \tag{77}$$

where

$$V^i(p) = \mathbb{E}\left[\sum_{j=t+2}^{+\infty} \beta^{j-t-2} \cdot \mathbb{I}\left(p_j \geqslant \bar{p}\right) \cdot s_j \,\middle|\, p, \theta = i\right], \ i = \{H, L\}.$$

Moreover, $V(p \mid p_t, b_t)$ is linear and strictly increasing in b_t for any given p.

Proof. Because $p \in \mathcal{E}(p_t)$ then the next consumer buys the product. Therefore (21) reduces to

$$V(p \mid p_t, b_t) = b_t (H - c) + (1 - b_t)(L - c) + \beta \left(b_t V^H(p) + (1 - b_t) V^L(p)\right),$$

which is (77). At the same time if $\theta = H$ then $\mathbb{I}(p_j \geqslant \bar{p}) \cdot s_j \geqslant 0$ for any j, while if $\theta = L$ then $\mathbb{I}(p_j \geqslant \bar{p}) \cdot s_j \leqslant 0$. Therefore $\left(H - c + \beta V^H(p)\right) > \left(L - c + \beta V^L(p)\right)$, and due to (77) $V(p_t, b_t \mid p)$ is thus strictly increasing in b_t. $\square$

Because $V(p \mid p_t, b_t)$ is linear in b_t, the higher b_t is, the more weight is assigned to $V^H(p)$ and less to $V^L(p)$. Therefore at a higher b_t a consumer would prefer to induce a posterior with higher $V^H(p)$ and lower $V^L(p)$. The next lemma shows it formally.

Lemma 17. *Suppose that $m', m'' \in \mathcal{M}(p_t)$ and $\bar{p} \leqslant q(p_t, m') < q(p_t, m'')$. Then either $V^H(q(p_t, m')) = V^H(q(p_t, m''))$ and $V^L(q(p_t, m')) = V^L(q(p_t, m''))$, or $V^H(q(p_t, m')) < V^H(q(p_t, m''))$ and $V^L(q(p_t, m')) > V^L(q(p_t, m''))$.*

Proof. If $V^H(q(p_t, m')) < V^H(q(p_t, m''))$ and $V^L(q(p_t, m')) < V^L(q(p_t, m''))$ then $V(q(p_t, m') \mid p_t, b_t) < V(q(p_t, m'') \mid p_t, b_t)$ and therefore $m' \notin \mathcal{M}(p_t)$, – a contradiction. Analogously, it can not be that $V^H(q(p_t, m')) > V^H(q(p_t, m''))$ and $V^L(q(p_t, m')) > V^L(q(p_t, m''))$. Finally, if for some p' and p'' we have $V^H(p')) > V^H(p''))$ and $V^L(p')) < V^L(p''))$ it implies that there exists $\bar{b} \in [0, 1]$ such that $V(p' \mid p_t, b_t) < V(p'' \mid p_t, b_t)$ for $b_t \in [0, \bar{b})$ and $V(p' \mid p_t, b_t) > V(p'' \mid p_t, b_t)$ for $b_t \in (\bar{b}, 1]$. Therefore we must have $p'' < p'$. $\square$

Corollary 3. *In any equilibrium for any p_t*

1. *$V^H(p)$ is a [weakly] increasing function of p on $\mathcal{E}(p_t)$,*

2. *$V^L(p)$ is a [weakly] decreasing function of p on $\mathcal{E}(p_t)$.*

Proof. Directly follows from Lemma 17. $\square$

Lemma 18. *At any public state p_t there exists $\bar{b}(p_t) \geqslant 0$ such that*

1. *If $r(m|p_t, b_t) > 0$ for some $b_t < \bar{b}_t$ then $q(p_t, m) \in \mathcal{S}(p_t)$;*

2. *If $r(m|p_t, b_t) > 0$ for some $b_t \geqslant \bar{b}_t$ then $q(p_t, m) \in \mathcal{E}(p_t)$;*

3. *If public state $p_\tau < 1$ with $\mathcal{S}(p_\tau) \neq \emptyset$ occurs with strictly positive probabilities $A^H > 0$ if $\theta = H$ and $A^L \in [0, A^H]$ if $\theta = L$ from public state p_t then $V(p_t, \bar{p}) > 0$;*

4. *$\bar{b}_t \leqslant \bar{p}$ and the equality is attained if and only if any chosen $p \in \mathcal{E}(p_t)$ leads to a continuation equilibrium where experimentation never stops, i.e. to an equilibrium which is payoff-equivalent to a babbling equilibria played from period $t+1$ onward.*

Proof. If $\mathcal{S}(p_t) = \emptyset$ then we can take $\bar{b}_t = 0$. Therefore we hereafter assume that $\mathcal{S}(p_t) \neq \emptyset$. To prove the first two claims assume the contrary. That is there exist $b'_t < b''_t$, m', m'' such that $q(p_t, m') \in \mathcal{E}(p_t)$, $q(p_t, m'') \in \mathcal{S}(p_t)$ and $r(m'|p_t, b'_t) > 0$, $r(m''|p_t, b''_t) > 0$. Then $V(p_t, b''_t) = 0$, and $V(p_t, b'_t) \geqslant 0$ as $\mathcal{S}(p_t) \neq \emptyset$. At the same time due to Lemma 16 we have

$$0 = V(p_t, b''_t) \geqslant V(q(p_t, m') \mid p_t, b''_t) > V(q(p_t, m') \mid p_t, b'_t) \geqslant 0,$$

which leads to a contradiction. This argument proves first two parts of the lemma.

By the first two parts it follows that if $\mathcal{S}(p_\tau) \neq \emptyset$ then $\bar{b}_\tau > 0$. Next, note that from point of view of consumer at p_t who holds private belief $b_t = \bar{p}$ at every future history stage payoff is not less then zero. Indeed, if there is no experimentation, then payoff is equal to 0, while if there is it is also equal to 0. Therefore

$$V(p_t, \bar{p}) \geqslant \beta^{\tau-t} \left[A^H \bar{p} \left(1 - F^H(\bar{s}_\tau)\right)(H - c) + A^L (1 - \bar{p}) \left(1 - F^L(\bar{s}_\tau)\right)(L - c)\right] > 0,$$

where $\bar{s}_\tau = l^{-1}\left(\ln\left(\frac{\bar{b}_\tau}{1-\bar{b}_\tau}\right) - \ln\left(\frac{p_\tau}{1-p_\tau}\right)\right)$, and l^{-1} is an inverse function to $\ln\left[\frac{f^H(s)}{f^L(s)}\right]$.

Finally, we prove the last part. If any posterior $p \in \mathcal{P}(p_t)$ induces babbling from $(t+1)$, then the current consumer decides whether all future consumers will buy the product or avoid it. Their expected utility from her point of view is then $\frac{1}{1-\beta}(\theta(b_t) - c)$ or zero in the two respective scenarios. Therefore as equilibrium is not babbling the current consumer makes all subsequent consumers buy the product if and only if $b_t \geqslant \bar{p}$.

To show the reverse statement it suffices to show that $V(p_t, \bar{p}) > 0$. Then by continuity it would imply $\bar{b}(p_t) < \bar{p}$. If at any public state p_τ which is a part of a history originating from p_t we have $\mathcal{S}(p_\tau) = \emptyset$, then this equilibrium is payoff-equivalent to a one where babbling equilibrium is played from period $t+1$ onwards. Therefore suppose there exists a public belief p_τ such that $\mathcal{S}(p_\tau) \neq \emptyset$. Denote the path of public beliefs that lead to this public belief as $p_{\tau-1}, p_{\tau-2}, \ldots$. Without loss also take minimal τ with such property. Consider public state $p_{\tau-1}$ such that $\mathcal{E}(p_{\tau-1}) = p_\tau$. Then by part 3 there exists $\tilde{p}_\tau$ such that $V(\tilde{p}_\tau \mid p_{\tau-1}, \bar{p}) = \bar{p}V^H(\tilde{p}_\tau) + (1-\bar{p})V^L(\tilde{p}_\tau) > 0$. Because $\mathcal{S}(p_{\tau-1}) = \emptyset$ as τ was taken to be minimal and by Corollary 3 and Lemma 17 we have that either $\tilde{p}_\tau$ or any other public belief with the same V^H and V^L is induced for all private beliefs $b_{\tau-1} \leqslant \bar{p}$ and by belief consistency in some neighborhood above $\bar{p}$. With respect to $G_{p_{\tau-1}}$ this region has a strictly positive measure. Therefore probability A^L that private belief $b_{\tau-1}$ is within this region if $\theta = L$ is positive and is lower than probability A^H that private belief $b_{\tau-1}$ is within this region if $\theta = H$. Therefore at $p_{\tau-2}$ there exists $\tilde{p}_{\tau-1} \in \mathcal{E}(p_{\tau-2})$ such that $V(\tilde{p}_{\tau-1} \mid p_{\tau-2}, \bar{p}) = \bar{p}V^H(\tilde{p}_{\tau-1}) + (1-\bar{p})V^L(\tilde{p}_{\tau-1}) \geqslant \beta A^H \bar{p} V^H(p_{\tau-1}) + \beta A^L (1-\bar{p}) V^L(p_{\tau-1}) > 0$. As τ is finite going backward through the history of public states we get the result for p_t which concludes the argument. □

Proof of Proposition 18. For decentralized case the proof directly follows from part 4 of Lemma 18. The proof for commitment case follows from Theorem 4. □

Lemma 19. *Suppose there exist $0 < l < \bar{p} < r < 1$, $\varepsilon > 0$ and $r(\varepsilon) > r$ such that*

$$\mathbb{E}_{x \sim G_p}\left[x \mid x \in [l, r]\right] = \mathbb{E}_{x \sim G_p}\left[x \mid x \in [l-\varepsilon, r(\varepsilon)]\right] = \bar{p}.$$

Then for any ε there exist $\underline{\gamma}, \overline{\gamma} > 0$ such that $\underline{\gamma}\varepsilon < r(\varepsilon) - r < \overline{\gamma}\varepsilon$.

Proof. Because both $f^L(x), f^H(x)$ are continuously differentiable there exist $\delta > 0$ and constants $0 < B_l < B_h$ such that $G'_p(x) \in (B_l, B_h)$ for all $x \in (\delta, 1-\delta)$. In what follows we consider this neighborhood. By definition

$$(G_p(r) - G_p(l)) \cdot \int_{l-\varepsilon}^{r(\varepsilon)} x dG_p(x) = (G_p(r(\varepsilon)) - G_p(l-\varepsilon)) \cdot \int_{l}^{r} x dG_p(x).$$

Rearranging terms we get

$$\int_{l-\varepsilon}^{l} (\bar{p} - x)\, dG_p(x) = \int_{r}^{r(\varepsilon)} (x - \bar{p})\, dG_p(x).$$

Using boundedness of $G_p'(x)$ we can therefore obtain

$$B_l\left(\bar{p}-l+\frac{\varepsilon}{2}\right)\cdot\varepsilon < B_h\left(r-\bar{p}+\frac{r(\varepsilon)-r}{2}\right)\cdot\left(r(\varepsilon)-r\right),$$
$$B_h\left(\bar{p}-l+\frac{\varepsilon}{2}\right)\cdot\varepsilon > B_l\left(r-\bar{p}+\frac{r(\varepsilon)-r}{2}\right)\cdot\left(r(\varepsilon)-r\right).$$

Solving these inequalities in $r(\varepsilon)-r$ we get

$$\frac{2B_l\left(\bar{p}-l+\frac{\varepsilon}{2}\right)}{\sqrt{2B_lB_h\left(\bar{p}-l+\frac{\varepsilon}{2}\right)+B_h^2\left(r-\bar{p}\right)^2}+B_h\left(r-\bar{p}\right)}\varepsilon < r(\varepsilon)-r < \frac{2B_h\left(\bar{p}-l+\frac{\varepsilon}{2}\right)}{\sqrt{2B_lB_h\left(\bar{p}-l+\frac{\varepsilon}{2}\right)+B_l^2\left(r-\bar{p}\right)^2}+B_l\left(r-\bar{p}\right)}\varepsilon.$$

Both the LHS and the RHS are continuous in ε and therefore attain their lowest and highest values respectively on $\varepsilon\in[0,1]$. Therefore we can find such constants $\underline{\gamma}$ and $\overline{\gamma}$ such that $\underline{\gamma}\varepsilon < r(\varepsilon)-r < \overline{\gamma}\varepsilon$. □

Lemma 20. *There exists $\delta>0$ such that for any p_t and any $p\in\mathcal{E}(p_t)$ we have $V^C(p\mid p_t,b_t)<0$ for all $b_t\in[0,\delta]$.*

Proof. Because $p\geqslant\bar{p}$ at time $t+1$ the next consumer certainly purchases the product and therefore by Lemma 16 we get

$$V^C(p\mid p_t,b_t)=\theta(b_t)-c+\beta\left(b_tV^{C,H}(p)+(1-b_t)V^{C,L}(p)\right).$$

Because $V^{C,H}(p)\leqslant\frac{H-c}{1-\beta}$ and $V^{C,L}(p)\leqslant 0$ we have that $V^C(p\mid p_t,b_t)\leqslant 0$ for all $b_t\leqslant\delta:=\frac{(1-\beta)(c-L)}{H-\beta c-(1-\beta)L}$. □

Lemma 21. *Consider public state p_t and $p',p''\in\mathcal{E}(p_t)$. Consider all public beliefs p'_τ and p''_τ that can be on paths originating from p' and p'' respectively and their associated messaging partitions $\Sigma(p'_\tau)$ and $\Sigma(p''_\tau)$. Then if $\Sigma(p''_\tau)$ is a consistent parallel shift of $\Sigma(p'_\tau)$ for each pair p'_τ, p''_τ by the same increment then $V^\theta(p')=V^\theta(p'')$ for $\theta\in\{H,L\}$.*

Proof. Note that probability that private belief b_t is lower than some threshold conditional on some public belief p_t depends only on $\ln\left(\frac{b_t}{1-b_t}\right)-\ln\left(\frac{p_t}{1-p_t}\right)$. If one shifts all the bounds for all sets in a given partition by the same increment, then expectations over all sets in the new partition shift by the same increment. Moreover, if the new partition is consistent with the previous one it implies therefore that conditional on θ under $\Sigma(p''_\tau)$ in any public belief p''_τ the experimentation stops with the same probability as it does under $\Sigma(p'_\tau)$ in p'_τ. Therefore $V^\theta(p')=V^\theta(p'')$. □

Proof of Theorem 4. We initially assume that at time t all consumers in the queue (current and future) can commit to a particular message structure. As the very initial step note that for every p_t an optimal $V^C(p_t)$ is achieved by choosing optimal $\Sigma(p_\tau)$ for every p_τ that can originate from p_t. Next, this partitions without loss can be assumed to be Markovian. That is $\Sigma(p_\tau)$ for all p_τ originating from p_t should not depend on t explicitly. Indeed, suppose there exists p and $\tau > t$ such that $p_t = p_\tau = p$, but optimal partitions originating from p_t and p_τ do not coincide. Then if $V^C(p_t) = V^C(p_\tau)$ without loss we can prescribe any of these partitions to p. If however, say, $V^C(p_t) < V^C(p_\tau)$ then we can prescribe partition corresponding to p_τ to p_t which then would strictly increase $V^C(p_t)$. Because partition for p_t was assumed to be optimal we get a contradiction. The case $V^C(p_t) > V^C(p_\tau)$ is analogous. Finally, in what follows we understand by $\hat{\Sigma}(p_t)$ a collection of optimal partitions for all p_τ with $\tau \geqslant t$ that deliver $V^C(p_t)$ (if it exists).

We next divide the proof into several steps outlined in the text.

Step 1. Note that ex ante value when public belief is p is equal to the value a consumer with *private* belief p gets when she induces public belief p. Formally, $V^C(p) = V^C(p \mid x, p)$ for any time-t public belief x. Indeed, public beliefs about the state coincide in these two cases and in the second case consumer's own belief is also equal to p and therefore she has the same expectations about future histories as a consumer who face public belief p.

Step 2. Take some p_t. There exists a set of public beliefs p_τ (including p_t) which can be reached on path originating from p_t. For each of them including p_t there exists an optimal partition $\Sigma(p_\tau)$. Note that for any $b_t > \bar{p}$ at any p_τ it is strictly optimal to induce experimentation. Indeed, suppose it is not, and at such b_t experimentation stops. Then we can exclude this belief from the set of private beliefs that induce no experimentation and induce babbling continuation in b_t forever after. In that case posterior for the pooled region where the experimentation stops will stay below $\bar{p}$, while for b_t we now get strictly positive continuation value.

We next show that for any p_t in an optimal partition $\Sigma(p_t)$ there exists $\delta > 0$ such that a consumer with private belief $b_t \in [0, \delta]$ sends a message that stops experimentation. Lemma 20 implies that if the consumer with private belief $b_t \leqslant \delta := \frac{(1-\beta)(c-L)}{H-\beta c-(1-\beta)L}$ induces public belief $p \in \mathcal{E}(p_t)$ then $V^C(p \mid p_t, b_t) < 0$. Suppose that in $\Sigma(p_t)$ there exists some set of points within $[0, \delta]$ that is pooled with private beliefs above $\bar{p}$ and the resulting posterior is above $\bar{p}$. Denote this posterior as p_{pool}. Then we can construct an alternative $\hat{\Sigma}(p_t)$ that will deliver at least the same ex ante value and where for all $b_t \in [0, \delta]$ the experimentation stops.

First, cut from the pooling region private beliefs below δ that induce experimentation and substitute continuation with no experimentation afterwards. If this set was of measure zero then we are done.[108] If it was of positive measure then we can cut the right end of this interval such that the resulting posterior stays on the level of p_{pool}. Now for all points below δ by Lemma 20 this adjustment provides an improvement. For all points that were not cut the value stays the same as induced posterior stayed the same. For the points that were cut from the right end we now need to prescribe continuation partitions that also provide an improvement. We do that by inducing truthful information transmission at all such points at p_t. In all resulting public states $p_{t+1} = b_t$ afterwards we use a parallel shift of $\hat{\Sigma}(p_t)$ by the increment of $\ln\left(\frac{b_t}{1-b_t}\right) - \ln\left(\frac{p_{pool}}{1-p_{pool}}\right)$.

We then without loss we can pool together all messages that induce no further experimentation into one message. We now know that resulting posterior of this region will be *uniformly* separated from $\bar{p}$ for any p_t.

Step 3. We show that $V^C(b_t \mid p_t, b_t)$ is a continuous, strictly increasing and [weakly] convex function of b_t for any p_t. Consider some b_t and denote as $\Sigma(b_t)$ the corresponding optimal partitions. Then for any point $\tilde{b}_t$ in the *right* neighborhood of b_t consider partitions $\hat{\Sigma}(\tilde{b}_t)$ that consist of shifted partitions $\hat{\Sigma}(b_t)$.[109] Because the posterior for the no experimentation region is uniformly separated from $\bar{p}$, in some neighborhood of b_t we can always do that in such a way that all the posteriors on path of play that were above $\bar{p}$ stay above it, and the ones that were below $\bar{p}$ stay below it. Then with $\hat{\Sigma}(\tilde{b}_t)$ we have $V^H(\tilde{b}_t) = V^H(b_t)$ and $V^L(\tilde{b}_t) = V^L(b_t)$.

Now suppose $V^C(b_t \mid p_t, b_t)$ is discontinuous at some $b_t \geqslant \bar{p}$. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$ we can find b'_t and b''_t with $|b''_t - b'_t| < \delta$ such that

$$V^C(b''_t \mid p_t, b''_t) - V^C(b'_t \mid p_t, b'_t) > \varepsilon.$$

For any p we than can take parallel shift of partitions $\hat{\Sigma}(b''_t)$ by the increment of $\ln\left(\frac{b'_t}{1-b'_t}\right) - \ln\left(\frac{b''_t}{1-b''_t}\right)$. There also exist such δ that this shift will be consistent, which by Lemma 21 implies that $V^\theta(b''_t) = V^\theta(b'_t)$. Because $V^H(p) - V^L(p) \leqslant H - L$ for any p we than have that taking $\delta < \frac{\varepsilon}{H-L}$ we get

$$V^C(b''_t \mid p_t, b''_t) - V^C(b'_t \mid p_t, b'_t) < \left(b''_t - b'_t\right) \cdot \left(V^H(p) - V^L(p)\right) = \varepsilon,$$

which gives us a contradiction.

Similarly, using parallel shifts and the fact that in some right neighborhood of any private belief

[108] Measure here is understood in terms of conditional measure on this pooling region.

[109] All by the same increment of $\ln\left(\frac{\tilde{b}_t}{1-\tilde{b}_t}\right) - \ln\left(\frac{b_t}{1-b_t}\right)$.

b_t they will be consistent, one can show strict monotonicity and convexity of $V(b_t \mid p_t, b_t)$.

Step 4. Because $V^C(p_t)$ is weakly convex so is $V^C(b_t \mid p_t, b_t)$ as a function of b_t. Therefore, first, one does not pool any private beliefs b_t above $\bar{p}$, and, second, if one pools beliefs below and above $\bar{p}$ induced posterior should be $\bar{p}$. Indeed, if one pools some private beliefs b_t below and above $\bar{p}$ and the resulting posterior is strictly above $\bar{p}$, then one can cut the right end of this region to lower induced public belief to $\bar{p}$. This weakly increases $V^C(p_t)$ by Jensen's inequality. Finally, the pooling region has to be convex below $\bar{p}$. For any $p \geqslant \bar{p}$, in particular $\bar{p}$ itself, we have

$$\left.\frac{dV^C(b_t \mid p_t, b_t)}{db_t}\right|_{b_t=\bar{p}} > 0$$

Therefore gains from pooling private beliefs below $\bar{p}$ decrease in distance from $\bar{p}$. At the same time due to convexity of $V^C(b_t \mid p_t, b_t)$ in b_t losses are increasing in distance from $\bar{p}$. Now suppose that the set of pooled private beliefs is not convex below $\bar{p}$. Then, first, we can cut private beliefs in the left end of the pooling region and attach the same measure of private beliefs directly to the left of the maximum pooling region around $\bar{p}$. This will weakly increase the posterior belief for the pooling region, which we can further decrease by cutting points from the right end of it. This will again improve $V^C(b_t \mid p_t, b_t)$ point-wise.

Therefore without loss we can assume that there exist $0 \leqslant l^C(p_t) \leqslant \bar{p}$ and set $I^C(p_t)$ with all it's points above $\bar{p}$ such that

1. for all $b_t \leqslant l(p_t)$ consumer sends message $m \in \mathcal{S}(p_t)$, i.e., experimentation stops.

2. for any $b_t \notin I^C(p_t)$ consumer truthfully transmits his private belief b_t (or what is the same his private payoff) to the public, that is $p_{t+1} = b_t$.

3. for all $b_t \in (l(p_t), p_t) \bigcup I^C(p_t)$ consumer sends message m such that $q(p_t, m) = \bar{p}$.

Step 5. We now show that it is always optimal to pool at least some private beliefs b_t around the cutoff for any p_t if $V^C(\bar{p} \mid p_t, \bar{p}) > 0$. In other words it has to be that $l(p_t) < \bar{p} < r(p_t)$. From the previous step we know that $V^C(\bar{p} \mid p_t, \bar{p}) > 0$, therefore pooling $\varepsilon \in (0, \delta)$ below $\bar{p}$ with some above $\bar{p}$ gives a benefit of $V^C(\bar{p} \mid p_t, \bar{p}) \cdot \varepsilon + \mathcal{O}(\varepsilon^2)$. At the same time because $V^C(b|p_t, b)$ is continuous in b, its slope is less than $H - L$, and by Lemma 19 losses associated with pooling beliefs above $\bar{p}$ do not exceed $B\varepsilon^2$ for some $B > 0$. Therefore there always exists such $\varepsilon > 0$ that $V^C(\bar{p} \mid p_t, \bar{p}) \cdot \varepsilon + \mathcal{O}(\varepsilon^2) > B\varepsilon^2$, and therefore pooling at least some beliefs around $\bar{p}$ is always optimal. If $V^C(\bar{p} \mid p_t, \bar{p}) = 0$ then it can only be the case if $V^C(b_t \mid p_t, b_t) = \theta(b_t) - c$ for $b_t \geqslant \bar{p}$.

Therefore for any p_t it is not optimal to pool any private beliefs around $\bar{p}$, and therefore in any p_t we have that no experimentation is induced for all $b_t \in [0, \bar{p}]$. Then if $A^\theta = \Pr(b_{t+1} \geqslant \bar{p} \mid \theta)$ we have

$$V^C(\bar{p} \mid p_t, \bar{p}) > \bar{p}A^H \cdot (H - c) + (1 - \bar{p})A^L \cdot (L - c) > 0,$$

because $A^H > A^L$. This contradicts $V^C(\bar{p} \mid p_t, \bar{p}) = 0$. Therefore $V^C(\bar{p} \mid p_t, \bar{p}) > 0$ and it is always optimal to pool at least some private beliefs around $\bar{p}$. □

Proof of Theorem 5. The first part of the Theorem follows from the fact that any decentralized solution induces partitions of private beliefs b_t for every p_t. The maximal value which can be achieved for any partitions in public state p_t is delivered by $V^C(p_t)$, and therefore $V^D(p_t) \leqslant V^C(p_t)$.

The second part follows from part 4 of Lemma 18. It implies that $\bar{b}_t < \bar{p}$. Therefore we can take $l^D(p_t) = \bar{b}_t$. By belief consistency and Lemma 17 there should exist $r^D(p_t) > \bar{p}$ and message $m \in \mathcal{M}$ such that m is sent for all $b_t \in [l^D(p_t), r^D(p_t)]$. □

Part IV

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Part V

Curriculum Vitae

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09/2014 – 07/2020	PhD studies at the Zurich Graduate School of Economics
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Professional Experience

09/2014 – 06/2019	Teaching assistant at the Department of Economics
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09/2013 – 08/2014	NES CSDSI, Research Fellow, Moscow, Russia.
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